POINTED AND COPOINTED HOPF ALGEBRAS AS COCYCLE DEFORMATIONS

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Abstract. We show that all finite dimensional pointed Hopf algebras with the same diagram in the classification scheme of Andruskiewitsch and Schneider are cocycle deformations of each other. This is done by giving first a suitable characterization of such Hopf algebras, which allows for the application of results by Masuoka about Morita-Takeuchi equivalence and by Schauenburg about Hopf Galois extensions. The "infinitesimal" part of the deforming cocycle and of the deformation determine the deformed multiplication and can be described explicitly in terms of Hochschild cohomology. Applications to, and results for copointed Hopf algebras are also considered.

0. Introduction

Finite dimensional pointed Hopf algebras over an algebraically closed field of characteristic zero, particularly when the group of points is abelian, have been studied quite extensively with various methods in [AS, BDG, Gr1, Mu]. The most far reaching results as yet in this area have been obtained in [AS], where a large class of such Hopf algebras are classified. In the present paper we will show, among other things, that all Hopf algebras in this class can be obtained by cocycle deformations. We also consider the "dual" case, where the Jacobson radical is a Hopf ideal. In the non-pointed case, in particular when the coradical is not a Hopf algebra, very little is known. Few examples occur of the literature [Ra2, Be], but no general description or classification results are available. It is the aim of this paper to contribute to the construction and classification of such Hopf algebras, in particular the copointed kind. By a copointed Hopf algebra we mean a Hopf algebra H whose Jacobson radical $\operatorname{Rad} H$ is a Hopf ideal and $H/\operatorname{Rad} H$ is a group algebra.

If H is a Hopf algebra with coradical a Hopf subalgebra then the graded coalgebra $\operatorname{gr}_c H$ associated with the coradical filtration is a graded Hopf algebra and its elements of positive degree form the radical. If the radical of H is a Hopf ideal then the graded algebra associated with the radical filtration is a graded Hopf algebra with $\operatorname{Cor}(\operatorname{gr}_a H) \cong H/\operatorname{Rad} H$. In either case we have $\operatorname{gr} H \cong R\#H_0$, where

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H_0 is the degree zero part and R is the braided Hopf algebra of coinvariants or invariants, respectively.

The Nichols algebra $B(V)$ of a crossed kG-module V is a connected graded braided Hopf algebra. $H(V) = B(V)\#kG$ is an ordinary graded Hopf algebra with coradical kG and the elements of positive degree form a Hopf ideal (the graded radical). A lifting of $H(V)$ is a pointed Hopf algebra H for which $\operatorname{gr}_c H \cong H(V)$. Such liftings are obtained by deforming the multiplication of $H(V)$. The lifting problem for V asks for the classification of all liftings of $H(V)$. This problem, together with the characterization of $B(V)$ and $H(V)$, have been solved by Andruskiewitsch and Schneider in [AS] for a large class of crossed kG-modules of finite Cartan type. It allows them to classify all finite dimensional pointed Hopf algebras A for which the order of the abelian group of points has no prime factors < 11. In this paper we find a description of these lifted Hopf algebras, which is suitable for the application of a result of Masuoka about Morita-Takeuchi equivalence [Ma] and of Schauenburg about Hopf Galois extensions [Sch], to prove that all liftings of a given $H(V)$ in this class are cocycle deformations of each other. As a result we see here that in the class of finite dimensional pointed Hopf algebras classified by Andruskiewitsch an Schneider [AS] all Hopf algebras H with isomorphic associated graded Hopf algebra $\operatorname{gr}_c H$ are monoidally Morita-Takeuchi equivalent, and therefore cocycle deformations of each other. For some special cases such results have been obtained in [Ma, Di]. Since $B(V)$ and $H(V)$ are graded the cocycles and deformations can be viewed in a formal setting. The infinitesimal parts are Hochschild cocycles. They determine the deformed multiplication and can be computed explicitly. The dual problem is to construct co-pointed Hopf algebras H by deforming the comultiplication of $H(V)$ in such a way that $\operatorname{gr}_a H \cong H(V)$, where $\operatorname{gr}_a H$ is the graded Hopf algebra associated with the radical filtration. Such 'liftings' can be viewed as cocycle deformations with convolution invertible coalgebra cocycles, which again can be discussed in a formal setting. In both cases the deformations are formal in the sense of [GS], the infinitesimal parts of these deformations determine the deformed multiplication and comultiplication, respectively, and are determined by the G-invariant part of the Hochschild cohomology of $B(V)$ or $H(V)$.

First we discuss what von Neumann regularity for an algebra and the "dual" concept of coregularity for a coalgebra entail in the case of a Hopf algebra. Of particular interest is the situation where the coradical is a regular Hopf algebra and/or $H/\operatorname{Rad} H$ is a coregular Hopf algebra. If both conditions are satisfied then $H \cong A\# \operatorname{Cor}(H)$, where $A = H^{co(\operatorname{Cor} H)}$ is the braided Hopf algebra of coinvariants over $\operatorname{Cor}(H)$. This happens in particular when $\operatorname{Cor}(H)$ is a finite dimensional Hopf subalgebra of H and $\operatorname{Rad}(H)$ is a Hopf ideal of finite codimension in H. It also happens for $\operatorname{gr}_c H$ when $H_0 = \operatorname{Cor}(H)$ is a regular Hopf algebra, and for $\operatorname{gr}_a H$ when $H_0 = H/\operatorname{Rad}(H)$ is a coregular Hopf algebra. A group Hopf algebra kG is always coregular, but it is regular if and only if every finitely generated

subgroup of G is finite. In Section 2 we continue with a short review of braided spaces, braided Hopf algebras, Nichols algebras and bosonization. in preparation for a useful characterization of the liftings for a large class of crossed modules over finite abelian groups in Section 3. With this characterization it is then possible to prove that liftings are Morita-Takeuchi equivalent (Section 3) by using Masuoka's pushout construction, and that they are cocycle deformations of each other (Section 4) by a result of Schauenburg. Cocycle deformations of multiplication and comultiplication, as well as their relation to Hochschild cohomology are discussed in Section 4. Some explicit examples are presented in Section 5, and duality of the two deformation procedures are explored in the final section.

It came to our attention that, in the preprint [Ma2] just posted in the archive, some special cases of the connection between 'liftings' and cocycle deformations are considered. After we had posted our paper A. Masuoka informed us that his Theorem 2 of [Ma] was missing a condition, as observed in [BDR]. For the verification of this additional condition, needed in our 3.5, we refer to the second version of [Ma2], where it now appears as an appendix.

1. Regularity and coregularity

An algebra A is (Von Neumann) regular if $a = axa$ has a solution for every $a \in A$. This is equivalent to saying that every left (right) A-module is flat, or also that every finitely generated left (right) ideal of A is generated by an idempotent [C, St], [We]. We say that a coalgebra C is coregular if every left (right) C-comodule is coflat.

Lemma 1.1. *If $C = \oplus_\nu C_\nu$ is a coalgebra and X a (right) C-comodule, then $X = \oplus X_\nu$, where $X_\nu = i_\nu^* X = X \otimes^C C_\nu$. Moreover, X is C-coflat if and only if X_ν is C_ν-coflat for every ν.*

Proposition 1.2. *The following properties of a coalgebra C are equivalent:*

- (a) *C is coregular.*
- (b) *Every subcoalgebra of C is coregular.*
- (c) *Every finite dimensional subcoalgebra of C is cosemisimple.*
- (d) $\mathrm{Cor}(C) = C$.
- (e) *C^* is a regular algebra.*

Proof. Let C be coregular and let D be a subcoalgebra of C. Every (right) D-comodule is also a (right) C-comodule, hence coflat as a C-comodule. Now, if X is a right D-comodule and $f\colon M \to N$ is a surjective D-comodule map, then in the commutative diagram

$$\begin{array}{ccc} X \otimes^D M & \longrightarrow & X \otimes^C M \\ \downarrow & & \downarrow \\ X \otimes^D N & \longrightarrow & X \otimes^C N \end{array}$$

the horizontal maps are bijective by the definition of the cotensor product, while the right-hand vertical map is surjective by the C coflatness of X, so that the left-hand vertical map is also surjective and X is therefore coflat as a D-module. Thus, every (right) D comodule is coflat and so D is coregular.

If D is a finite dimensional subcoalgebra of C, which is coregular, then D^* is a finite dimensional regular algebra, hence semisimple, so that D is cosemisimple. (A module is flat whenever every finitely generated submodule is flat [St]. An A-module Y is flat if and only if $Y \otimes_R I \to Y$ is injective for every finitely generated left ideal I of A [C, St]. Thus, it suffices to consider finitely generated modules, in which case $(X \otimes^D M)^* \cong X^* \otimes_{D^*} M^*$.)

If every finite dimensional subcoalgebra of C is cosemisimple then $\mathrm{Cor}(C) = C$, since every element of C is contained in a finite dimensional subcoalgebra.

That (d) implies (a) follows from Lemma 1.1. It remains to show that (d) is equivalent to (e). If $C = \mathrm{Cor}(C)$ then C^* is a product of finite dimensional simple, and hence regular algebras. But a product of algebras is regular if and only if each factor is regular, so that C^* is regular. Conversely, if C is not coregular, then it contains a finite dimensional subcoalgebra D which is not cosemisimple, and D^* is a finite dimensional non-regular quotient algebra of C^*, so that C^* is not regular. □

Proposition 1.3. *If A is a regular algebra then:*

(a) $\mathrm{Rad}(A) = 0$.
(b) *Every quotient algebra of A is regular.*
(c) *Every finite dimensional quotient algebra of A is semisimple.*
(d) A° *is a coregular coalgebra, i.e.* $\mathrm{Cor}(A^\circ) = A^\circ$.

Proof. If $a \in \mathrm{Rad}(A)$ then $a = axa$ for some $x \in A$ implies that $a(1 - xa) = 0$ and hence $a = 0$, since $1 - xa$ is invertible.

If $a = axa$ in A then $\bar{a}\bar{x}\bar{a} = \bar{a}$ in A/I for any ideal I of A. If A/I is finite dimensional and regular then it is semisimple, since $\mathrm{Rad}(A) = 0$.

$A^\circ = \mathrm{colim}(A/I)^*$, where the colimit is over all cofinite ideals of A. If A is regular and I is a cofinite ideal then A/I is semisimple and $(A/I)^*$ is cosemisimple, so that $\mathrm{Cor}(A^\circ) = A^\circ$. Assertion (d) also appears as Proposition 3.2 in [Cu]. □

Lemma 1.4. *If A is a Von Neumann regular subring of the ring B then* $A \cap \mathrm{Rad}\, B = 0$.

Proof. If $a \in A \cap \mathrm{Rad}\, B$ then $a = axa$ has a solution in A, say $x = a'$, since A is Von Neumann regular, and $1 - a'a$ is invertible, since $a \in \mathrm{Rad}(B)$. But then $a(1 - a'a) = 0$ implies that $a = 0$. □

The following example shows that the conclusion of this Lemma does not hold in general when A is not regular. If the polynomial algebra $B = k[x]$ is considered in the usual way as a subalgebra of the power series algebra $A = k[[x]]$ then $\mathrm{Rad}(B) = Bx$ and $A \cap \mathrm{Rad}(B) = Ax$, but $\mathrm{Rad}(A) = 0$.

Proposition 1.5. *Let H be a Hopf algebra.*

(a) *If $\operatorname{Cor}(H)$ is a Von Neumann regular Hopf subalgebra of H, in particular if $\operatorname{Cor}(H)$ is a finite dimensional Hopf subalgebra of H, then the algebra map $\operatorname{Cor}(H) \to H \to H/\operatorname{Rad}(H)$ is injective.*

(b) *If $\operatorname{Rad}(H)$ is a Hopf ideal and $H/\operatorname{Rad}(H)$ is coregular, in particular if $\operatorname{Rad}(H)$ is a Hopf ideal of finite codimension in H, then the coalgebra map $\operatorname{Cor}(H) \to H \to H/\operatorname{Rad}(H)$ is surjective.*

(c) *If $\operatorname{Cor}(H)$ is a von Neumann regular Hopf subalgebra of H and $\operatorname{Rad}(H)$ a Hopf ideal with $H/\operatorname{Rad}(H)$ coregular then $\operatorname{Cor}(H) \to H \to H/\operatorname{Rad}(H)$ is a Hopf algebra isomorphism and $H \cong A \rtimes \operatorname{Cor}(H)$, where $A = H^{co(\operatorname{Cor} H)}$ is the braided Hopf algebra of coinvariants over $\operatorname{Cor}(H)$. This happens in particular when $\operatorname{Cor}(H)$ is a finite dimensional Hopf subalgebra of H and $\operatorname{Rad}(H)$ is a Hopf ideal of finite codimension in H.*

Proof. a) If $\operatorname{Cor}(H)$ is a Von Neumann regular Hopf subalgebra of H then for any $a \in \operatorname{Cor}(H) \cap \operatorname{Rad}(H) \subset \operatorname{Cor}(H)$ the equation $a = axa$ has a solution in $\operatorname{Cor}(H)$ so that $a(1 - xa) = 0$. Since $1 - xa$ is invertible in H it follows that $a = 0$, and hence $\operatorname{Cor}(H) \cap \operatorname{Rad}(H) = 0$. In particular, if $\operatorname{Cor}(H)$ is a finite dimensional Hopf subalgebra of H then it is cosemisimple, hence semisimple by [LR], and thus Von Neumann regular, so that $\operatorname{Cor}(H) \cap \operatorname{Rad}(H) \subseteq \operatorname{Rad}(\operatorname{Cor}(H)) = 0$ by Lemma 1.4

b) If $\operatorname{Rad}(H)$ is a Hopf ideal in H and $H/\operatorname{Rad}(H)$ is a coregular Hopf algebra, then $\operatorname{Cor}(H/\operatorname{Rad}(H)) = H/\operatorname{Rad}(H)$. Moreover, since any surjective coalgebra map $\eta\colon C \to D$, where $D = \operatorname{Cor}(D)$, maps $\operatorname{Cor} C$ onto $\operatorname{Cor} D$ [Mo, Corollary 5.3.5] it follows that the canonical map $\operatorname{Cor}(H) \to H \to H/\operatorname{Rad}(H)$ is surjective. In particular when $\operatorname{Rad}(H)$ is a Hopf ideal of finite codimension in H then $H/\operatorname{Rad}(H)$ is a semisimple Hopf algebra, hence cosemisimple by [LR].

c) It follows directly from a) and b) that $\pi\colon H \to H/\operatorname{Rad}(H)$ is a Hopf algebra with projection. Now apply [Ra1]. □

Pointed and copointed Hopf algebras. A Hopf algebra H is pointed if its coradical $\operatorname{Cor} H$ is equal to the group algebra of the group of points $G(H)$. In this case the coradical filtration is an ascending Hopf algebra filtration and the associated graded Hopf algebra $\operatorname{gr}^c H$ has the obvious injection $\kappa^c\colon kG \to \operatorname{gr}^c H$ and projection $\pi^c\colon \operatorname{gr}^c H \to kG$ such that $\pi^c\kappa^c = 1$.

We say that H is copointed if its radical $\operatorname{Rad} H$ is a Hopf ideal and $H/\operatorname{Rad} H$ is a group algebra kG. Here the radical filtration is an descending Hopf algebra filtration and again the associated graded Hopf algebra $\operatorname{gr}^r H$ has the obvious projection $\pi^r\colon \operatorname{gr}^r H \to kG$ and an injection $\kappa^r\colon kG \to \operatorname{gr}^r H$ such that $\pi^r\kappa^r = 1$.

In both cases above $\operatorname{gr} H$ is graded, pointed and copointed, and by [Ra1] $\operatorname{gr} H \cong A\#kG$, where $A = \{x \in H \mid (\pi \otimes 1)\Delta(x) = 1 \otimes x\}$ is the graded connected braided Hopf algebra of coinvariants.

Lemma 1.6. *If the Hopf algebra H is pointed and copointed then $\operatorname{Cor} H \cong H/\operatorname{Rad} H$ and H is a Hopf algebra with projection. Moreover, $R\#kG \cong H.$,*

where $R = \{ x \in \operatorname{gr} H |\, (p \otimes 1)\Delta(x) = 1 \otimes x\}$ *is the connected braided Hopf algebra of coinvariants of* H*. This is the case in particular for* $\operatorname{gr}^c H$ *and for* $\operatorname{gr}^r H$ *when* H *is pointed or copointed, respectively.*

Proof. A surjective coalgebra map $\eta\colon C \to D$, where $D = \operatorname{Cor}(D)$, maps $\operatorname{Cor} C$ onto $\operatorname{Cor} D$ [Mo]. Thus, the composite $\operatorname{Cor} H \to H \to H/\operatorname{Rad} H$ is a bijection. The isomorphism is that of [Ra1]. □

2. Braided Hopf algebras and the (bi-)cross product

A braided monoidal category $\mathcal{V}$ is a monoidal category together with a natural morphism $c\colon V \otimes W \to W \otimes V$ such that

(1) $c_{k,V} = \tau = c_{V,k}$,
(2) $c_{U\otimes V,W} = (c_{U,W} \otimes 1)(1 \otimes c_{V,W})$,
(3) $c_{U,V\otimes W} = (1 \otimes c_{U,W})(c_{U,V} \otimes 1)$,
(4) $c(f \otimes g) = (g \otimes f)c$.

Braided algebras, braided coalgebras and braided Hopf algebras are now defined with this tensor product and braiding in mind. The compatibility condition $\Delta\, \mathrm{m} = (m\otimes m)(1\otimes c\otimes 1)(\Delta\otimes\Delta)$ between multiplication and comultiplication in a braided Hopf algebra A involves the braiding $c\colon A \otimes A \to A \otimes A$, so that the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "(1\otimes c\otimes 1)(\Delta\otimes\Delta)"] \arrow[d, "m"'] & A \otimes A \otimes A \otimes A \arrow[d, "m\otimes m"'] \\ A \arrow[r, "\Delta"] & A \otimes A \end{tikzcd}$$

commutes, i.e. multiplication and unit are morphisms of braided coalgebras or, equivalently, comultiplication and counit are maps of braided algebras.

2.1. **Primitives and indecomposables.** The vector space of primitives

$$P(A) = \{ y \in A |\, \Delta(y) = y \otimes 1 + 1 \otimes y\} \cong \ker(\tilde{\Delta})\colon A/k \to A/k \otimes A/k$$

of a braided Hopf algebra A is a braided vector space, since Δ is a map in $\mathcal{V}$. The c-bracket map $[-,-]_c = m(1 \otimes 1 - c)\colon A \otimes A \to A$ restricted to $P(A)$ satisfies $\Delta[x,y]_c = [x,y]_c \otimes 1 + (1 - c^2)x \otimes y + 1 \otimes [x,y]_c$; in particular $[x,y]_c \in P(A)$ if and only $c^2(x \otimes y) = x \otimes y$. Moreover, if $x \in P(A)$ and $c(x \otimes x) = qx \otimes x$ then $\Delta x^n = \sum_{i+j=n} \binom{n}{i}_q x^i \otimes x^j$, where $\binom{n}{i}_q = \frac{n_q!}{i_q!(n-i)_q!}$ are the q-binomial coefficients (the Gauss polynomials) for q, $m_q! = 1_q 2_q ... m_q$ with $j_q = 1 + q + ... + q^{j-1}$ if $j > 0$ and $0_q! = 1$. If $q = 1$ then $j_q = j$ and we have the ordinary binomial coefficients, otherwise $j_q = \frac{1-q^j}{1-q}$. In particular, if q has order n then $\binom{n}{i}_q = 0$ for $0 < i < n$, and hence $x^n \in P(A)$.

Lemma 2.1. *Let* $\{x_i\}$ *be a basis of* $P(A)$ *such that* $c(x_i \otimes x_j) = q_{ji} x_j x_i$*. If* $q_{ji} q_{ij} q_{ii}^{r-1} = 1$ *then* $\operatorname{ad} x_i^r(x_j)$ *is primitive.*

Proof. See for example [AS1] appendix 1. □

The vector space of indecomposables

$$Q(A) = JA/JA^2 = \mathrm{cok}(\tilde{m}\colon JA \otimes JA \to JA),$$

where $JA = \ker(\epsilon)$, is in $\mathcal{V}_c$. The c-cobracket map $\delta_c = (1 \otimes 1 - c)\Delta\colon A \to A \otimes A$ restricts to JA, since

$$\Delta(\bar{a}) = \bar{a} \otimes 1 + 1 \otimes \bar{a} + \sum \bar{a}_i \otimes \bar{b}_i$$

and hence

$$\delta(\bar{a}) = \sum (\bar{a}_i \otimes \bar{b}_i - c(\bar{a}_i \otimes \bar{b}_i))$$

is in $JA \otimes JA$ for every $\bar{a} = a - \epsilon(a) \in JA$. Moreover,

$$\delta(\bar{a}\bar{b}) - (\bar{a} \otimes \bar{b} - c^2(\bar{a} \otimes \bar{b}))$$

is in $JA^2 \otimes JA + JA \otimes JA^2$. In particular, if $c^2(\bar{a} \otimes \bar{b}) = \bar{a} \otimes \bar{b}$, then $\delta(\bar{a} \otimes \bar{b}) \in JA^2 \otimes JA + JA \otimes JA^2$.

2.2. **The free and the cofree graded braided Hopf algebras.** The forgetful functor $U\colon \mathrm{Alg}_c \to \mathcal{V}_c$ has a left-adjoint $\mathcal{A}\colon \mathcal{V}_c \to \mathrm{Alg}_c$ and the forgetful functor $U\colon \mathrm{Coalg}_c \to \mathcal{V}_c$ has a right-adjoint $\mathcal{C}\colon \mathcal{V}_c \to \mathrm{Coalg}_c$, the free braided graded algebra functor and the cofree graded braided coalgebra functor, respectively. Moreover, there is a natural transformation $\mathcal{S}\colon \mathcal{A} \to \mathcal{C}$, the shuffle map or quantum symmetrizer. They can be described as follows.

If (V, μ, δ) is a braided vector space then the tensor powers $T_0(V) = k$, $T_{n+1}(V) = V \otimes T_n(V)$ are braided vector spaces as well and so is $T(V) = \oplus_n T_n(V)$. The ordinary tensor algebra structure makes $T(V)$ the free connected graded braided algebra, and the ordinary tensor coalgebra structure makes it the cofree connected graded braided coalgebra.

By the universal property of the graded braided tensor algebra $T(V)$ the linear map $\Delta_1 = \mathrm{incl}\,\mathrm{diag}\colon V \to T(V) \otimes T(V)$, $\Delta_1(v) = v \otimes 1 + 1 \otimes v$, induces the c-shuffle comultiplication $\Delta_{\mathcal{A}}\colon T(V) \to T(V) \otimes T(V)$, which is a homomorphism of braided algebras, so that $\Delta_{\mathcal{A}} m = (m \otimes m)(1 \otimes c \otimes 1)(\Delta_{\mathcal{A}} \otimes \Delta_{\mathcal{A}})$. Moreover, the linear map $s_1\colon V \to T(V)$, $s_1(v) = -v$, extends uniquely to a c-antipode $s_{\mathcal{A}}\colon T(V) \to T(V)$, such that $s_{\mathcal{A}} m = m(s_{\mathcal{A}} \otimes s_{\mathcal{A}})c$, $\Delta_{\mathcal{A}} s_{\mathcal{A}} = c(s_{\mathcal{A}} \otimes s_{\mathcal{A}})\Delta_{\mathcal{A}}$ and $m(1 \otimes s_{\mathcal{A}})\Delta_{\mathcal{A}} = \iota\epsilon = m(s_{\mathcal{A}} \otimes 1)\Delta_{\mathcal{A}}$, thus making $\mathcal{A}(V) = (T(V), m, \Delta_{\mathcal{A}}, s_{\mathcal{A}})$ the free connected graded braided Hopf algebra. This defines a functor $\mathcal{A}\colon \mathcal{V}_c \to Hopf_c$, left-adjoint to the space of primitives functor $P\colon Hopf_c \to \mathcal{V}_c$.

On the other hand, by the universal property of the cofree connected graded braided coalgebra $T(V)$ there is a unique c-shuffle multiplication $m_{\mathcal{C}}\colon T(V) \otimes T(V) \to T(V)$, which is the homomorphism of braided coalgebras induced by the linear map $m_1 = +\,\mathrm{proj}\colon T(V) \otimes T(V) \to V$, so that $\Delta m_{\mathcal{C}} = (m_{\mathcal{C}} \otimes m_{\mathcal{C}})(1 \otimes c \otimes 1)(\Delta \otimes \Delta)$. The linear map $s_1 = -\,\mathrm{proj}\colon T(V) \to V$ induces uniquely a c-antipode $s_{\mathcal{C}}\colon T(V) \to T(V)$, such that $s_{\mathcal{C}} m_{\mathcal{C}} = m_{\mathcal{C}}(s_{\mathcal{C}} \otimes s_{\mathcal{C}})c$, $\Delta s_{\mathcal{C}} = c(s_{\mathcal{C}} \otimes s_{\mathcal{C}})\Delta$ and $m_{\mathcal{C}}(1 \otimes s_{\mathcal{C}})\Delta = \iota\epsilon = m_{\mathcal{C}}(s_{\mathcal{C}} \otimes 1)\Delta$, making $\mathcal{C}(V) = (T(V), \Delta, m_{\mathcal{C}}, s_{\mathcal{C}})$ the

cofree connected graded braided Hopf algebra. The functor $\mathcal{C}\colon \mathcal{V}_c \to Hopf_c$ is right-adjoint to the space of indecomposables functor $Q\colon Hopf_c \to \mathcal{V}_c$.

The adjunctions just described provide natural isomorphisms

$$\mathcal{V}_c(Q\mathcal{A}(V), V) \cong Hopf_c(\mathcal{A}(V), \mathcal{C}(V)) \cong \mathcal{V}_c(V, P\mathcal{C}(V)),$$

and by construction we also have natural isomorphisms

$$Q\mathcal{A}(V) \cong V \quad , \quad V \cong P\mathcal{C}(V).$$

The resulting natural isomorphism

$$Vect_c(V, V) \cong Hopf_c(\mathcal{A}(V), \mathcal{C}(V))$$

sends the identity morphism of V to the quantum symmetrizer $\mathcal{S}\colon \mathcal{A}(V) \to \mathcal{C}(V)$. The image of $\mathcal{S}$ is the Nichols algebra

$$\mathcal{B}(V) \cong \mathcal{A}(V)/\ker(\mathcal{S}) \cong \operatorname{im}\mathcal{S} \subset \mathcal{C}(V)$$

and $Q\mathcal{B}(V) \cong V \cong P\mathcal{B}(V)$.

An explicit description of the quantum symmetrizer can be obtained in term of the action of the braid groups B_n on the tensor powers $V^{\otimes n}$ as follows. The Braid group B_n can be defined by generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

The symmetric group S_n is obtained by imposing the additional relations

(1) $\sigma_i^2 = 1$ for $i = 1, 2, \ldots, n-1$.

If we denote the corresponding generators of S_n by $\tau_1, \tau_2, \ldots, \tau_{n-1}$, then the kernel of the canonical quotient map $f\colon B_n \to S_n$ given by $f(\sigma_i) = \tau_i$ is the normal subgroup of B_n generated by the squares $\sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2$. The set theoretic section $u\colon S_n \to B_n$ defined by $u(\tau_{i_1}\tau_{i_2}\cdots\tau_{i_l}) = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_l}$ for any reduced word $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_l}$ of S_n is called the Matsumoto section. If $l(\tau\tau') = l(\tau) + l(\tau')$ then $u(\tau\tau') = u(\tau)u(\tau")$. The element $\mathcal{S} = \sum_{\tau\in S_n} u(\tau)$ of kB_n is called the quantum symmetrizer.

The braiding map $c\colon V\otimes V \to V\otimes V$ induces a linear representation $\rho_n\colon B_n \to \operatorname{Aut}(V^{\otimes n})$ by $\rho_n(\sigma_i) = 1^{\otimes(i-1)} \otimes c \otimes 1^{\otimes(n-i-1)}$ for every $n \geq 0$, and hence a graded linear map $\mathcal{S}\colon T(V) \to T(V)$. If $X_{i,j}$ is the subset of (i,j)-shuffles in S_n then $\mathcal{S}_{i,j} = \sum_{\tau\in X_{i,j}} u(\tau)$ is an element of kB_n, and $\Delta_{\mathcal{A}}(n) = \sum_{i+j=n} \mathcal{S}_{i,j}$ and $m_{\mathcal{C}}(i,j) = \mathcal{S}_{i,j}$. Moreover, since $\mathcal{S}_{i,j}(\mathcal{S}_i \otimes \mathcal{S}_j) = \mathcal{S}_n$ whenever $r+s=n$, it follows that the quantum symmetrizer actually induces homomorphism of graded braided Hopf algebras

$$\mathcal{S}\colon \mathcal{A}(V) \to \mathcal{C}(V),$$

also called quantum symmetrizer.

The free graded braided Hopf algebra $\mathcal{A}(V)$ is the graded braided tensoralgebra $T(V)$, with the graded braided c-shuffle comultiplication $\Delta_{\mathcal{A}}\colon \mathcal{A}(V) \to \mathcal{A}(V) \otimes \mathcal{A}(V)$ and antipode $s_{\mathcal{A}}\colon \mathcal{A}(V) \to \mathcal{A}(V)$, induced by the universal property of

$T(V)$ from the natural maps $\Delta_1\colon V \to T(V) \otimes T(V)$, $\Delta_1(v) = v \otimes 1 + 1 \otimes v$, and $s_1\colon V \to T(V)$, $s_1(v) = -v$, respectively.

$$\begin{array}{ccc} \mathcal{A}(V) \otimes \mathcal{A}(V) & \xrightarrow{(1\otimes c\otimes 1)(\Delta_{sh}\otimes\Delta_{sh})} & \mathcal{A}(V) \otimes \mathcal{A}(V) \otimes \mathcal{A}(V) \otimes \mathcal{A}(V) \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m\otimes m} \\ \mathcal{A}(V) & \xrightarrow{\Delta_{sh}} & \mathcal{A}(V) \otimes \mathcal{A}(V) \end{array}$$

commute. This defines a functor $\mathcal{A}\colon \mathcal{V}_c \to Hopf_c$, left-adjoint to the space of primitives functor $P\colon Hopf_c \to \mathcal{V}_c$.

The cofree graded braided Hopf algebra $\mathcal{C}(V)$ is the graded braided tensor coalgebra with the graded braided tensor coalgebra structure induced by the canonical decompositions $\Delta_{i,j}\colon T_n(V) \to T_i(V) \otimes T_j(V)$ for $n = i+j$, and the graded braided c-shuffle multiplication induced by the c-shuffles $m_{i,j}\colon T_i(V) \otimes T_j(V) \to T_{i+j}(V)$, which make the diagram

$$\begin{array}{ccc} \mathcal{C}(V) \otimes \mathcal{C}(V) & \xrightarrow{(1\otimes c\otimes 1)(\Delta\otimes\Delta)} & \mathcal{C}(V) \otimes \mathcal{C}(V) \otimes \mathcal{C}(V) \otimes \mathcal{C}(V) \\ {\scriptstyle m_{sh}}\big\downarrow & & \big\downarrow{\scriptstyle m_{sh}\otimes m_{sh}} \\ \mathcal{C}(V) & \xrightarrow{\Delta} & \mathcal{C}(V) \otimes \mathcal{C}(V) \end{array}$$

commute. The functor $\mathcal{C}\colon \mathcal{V}_c \to Hopf_c$ is right-adjoint to the space of indecomposables functor $Q\colon Hopf_c \to \mathcal{V}_c$.

2.3. **Crossed modules.** A prime example of a braided monoidal category is the category of crossed H-modules YD_H^H for a Hopf algebra H. A crossed H-module or a Yetter-Drinfel'd H-module, (V, μ, δ) is a vector space V with a H-module structure $\mu\colon H \otimes V \to V$, $\mu(h \otimes v) = hv$, and a H-comodule structure $\delta\colon V \to H \otimes V$, $\delta(v) = v_{-1} \otimes v_0$, such that $h\delta(v) = h_1 v_{-1} \otimes h_2 v_0 = (h_1 v)_{-1} h_2 \otimes (h_1 v)_0$, or

$$(m \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes \delta) = (m \otimes 1)(1 \otimes \tau)(\delta\mu \otimes 1)(1 \otimes \tau)(\Delta \otimes 1),$$

i.e such that the diagram

$$\begin{array}{ccccc} H \otimes V & \xrightarrow{\Delta\otimes\delta} & H \otimes H \otimes H \otimes V & \xrightarrow{1\otimes\tau\otimes 1} & H \otimes H \otimes H \otimes V \\ {\scriptstyle (1\otimes\tau)(\Delta\otimes 1)}\big\downarrow & & & & \big\downarrow{\scriptstyle m\otimes\mu} \\ H \otimes V \otimes H & \xrightarrow{\delta\mu\otimes 1} & H \otimes V \otimes H & \xrightarrow{(m\otimes 1)(1\otimes\tau)} & H \otimes V \end{array}$$

commutes. This is the case in particular when $\delta(hv) = h_1 v_{-1} s(h_3) \otimes h_2 v_0$, i.e: when the diagram

$$\begin{array}{ccc} H\otimes V & \xrightarrow{(1\otimes\phi\otimes 1)(\Delta\otimes\delta)} & H\otimes H\otimes H\otimes V \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle m\otimes\mu} \\ V & \xrightarrow{\delta} & H\otimes V \end{array}$$

commutes, where $\phi = (m\otimes 1)(1\otimes s\otimes 1)(1\otimes\tau\Delta)\tau\colon H\otimes H\to H\otimes H$, $\phi(g\otimes h) = hs(g_2)\otimes g_1$. The braided H-modules with the obvious homomorphisms form a braided monoidal category, with the ordinary tensor product of vector spaces together with diagonal action and diagonal coaction. The braiding, given by $c(v\otimes w) = v_{-1}(w)\otimes v_0$,

$$c = (\mu\otimes 1)(1\otimes\tau)(\delta\otimes 1)\colon V\otimes W\to W\otimes V,$$

clearly satisfies the braiding conditions. The crossed H-module $(k, \mu = \varepsilon\otimes 1, \delta = \iota\otimes 1)$ acts as a unit for the tensor. Moreover, $(H, \mathrm{adj}, \Delta)$ and (H, m, coadj) are crossed H-modules, where $\mathrm{adj}(h\otimes h') = h_1h'S(h_2)$ and $\mathrm{coadj}(h) = h_1S(h_3)\otimes h_2$.

2.4. **The pushout construction for bi-cross products.** Recall Masuoka's pushout construction for Hopf algebras [Ma], [Gr1]. If A is a Hopf algebra then $\mathrm{Alg}(A,k)$ s a group under convolution which acts on A by conjugation as Hopf algebra automorphisms.

Lemma 2.2. *For every Hopf algebra A the group $\mathrm{Alg}(A,k)$ acts on A by conjugation' as Hopf algebra automorphisms*

$$\rho\colon\ \mathrm{Alg}(A,k)\to\mathrm{Aut}_{Hopf}(A),$$

where $\rho_f = f*1*fs$, i.e: $\rho_f(x) = f(x_1)x_2f(sx_3)$. The image of ρ is a normal subgroup of $\mathrm{Aut}_{Hopf}(A)$.

Proof. It is easy to verify that ρ_f is an Hopf algebra map. The definition of ρ shows that $\rho_{f_1*f_2} = f_1*f_2*1*f_2s*f_1s = \rho_{f_1}\rho_{f_2}$ and, since $f*fs = \varepsilon = fs*f$, it follows that $\rho_f\rho_{fs} = 1 = \rho_{fs}\rho_f$, so that ρ_f is a Hopf algebra automorphism. If $\phi\in\mathrm{Aut}_{Hopf}(A)$ and $f\in\mathrm{Alg}(A,k)$ then $\phi^{-1}\rho_f\phi = f\phi*1*fs\phi = \rho_{f\phi}$, hence the image of ρ is a normal subgroup. □

Two Hopf ideals I and J of A are said to be conjugate if $J = \rho_f(I) = f*I*fs$ for some $f\in\mathrm{Alg}(A,k)$. If $x\in P_{1,g}$ is a $(1,g)$-primitive then

$$\rho_f(x) = f(x) + f(g)x + f(g)gfs(x) = f(g)x + f(x)(g-1)].$$

Theorem 2.3. [Ma, Theorem 2][BDR, Theorem 3.4] *Let A' be a Hopf subalgebra of A. If the Hopf ideals I and J of A' are conjugate and $A/(f*I)\neq 0$ then the quotient Hopf algebras $A/(I)$ and $A/(J)$ by the Hopf ideals in A generated by I and J are monoidally Morita-Takeuchi equivalent, i.e: there exists a k-linear monoidal equivalence between their (left) comodule categories.*

Proof. With the correction that $A/(f*I)\neq 0$ ([BDR], Theorem 3.4), Masuoka's result ([Ma], Theorem 2) that there is a $(A/(I),A/(J))$-biGalois object, namely $A/(f*I)$, holds , and we can invoke [Sch, Corollary 5.7], to see that $A/(I)$ and $A/(J)$ are Morita-Takeuchi equivalent. □

Observe, as Masuoka did [Ma], that the commutative square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ B/I & \longrightarrow & A/(I) \end{array}$$

is a pushout of Hopf algebras.

If R is a braided Hopf algebra in the braided category of crossed H-modules then the bi-cross product $R\#H$ is an ordinary Hopf algebra with multiplication

$$(x\#h)(x'\#h') = xh_1(x')\#h_2h'$$

and comultiplication

$$\Delta(x\#h) = x_1\#(x_2)_{-1}h_1\otimes(x_2)_0\#h_2.$$

The (left) action of H on R induces a (right) action on $\mathrm{Alg}(R,k)$ by $fh(x)=f(hx)$. An algebra map $f\colon R\to k$ is H-invariant if $fh=\varepsilon(h)f$ for all $h\in H$.

Proposition 2.4. *Let K be a Hopf algebra in the braided category of crossed H-modules and let $\mathrm{Alg}_H(K,k)$ be the set of H-invariant algebra maps. Then:*

1. $\mathrm{Alg}_H(K,k)$ *is a group under convolution.*
2. *The restriction map* $\mathrm{res}\colon {}_H\mathrm{Alg}_H(K\#H,k)\to\mathrm{Alg}_H(K,k)$, $\mathrm{res}(F)=F\otimes\iota$, *is an isomorphism of groups with inverse given by* $\mathrm{res}^{-1}(f)=f\otimes\varepsilon$.
3. *The image of the conjugation homomorphism*

$$\Theta=\rho\,\mathrm{res}^{-1}\colon\ \mathrm{Alg}_H(K,k)\to\mathrm{Aut}_{Hopf}(K\#H)$$

is contained in $\widetilde{\mathrm{Aut}}_{Hopf}(K\#H)=\big\{\phi\in\mathrm{Aut}_{Hopf}(K\#H)|\,\phi_{|H}=id\big\}$.

Proof. The set of algebra maps $\mathrm{Alg}(K,k)$ may not be a group, but since the coequalizer $K^H=\mathrm{coeq}(\mu,\varepsilon\otimes 1\colon H\otimes K\to K$ is an ordinary Hopf algebra, $\mathrm{Alg}_H(K,k)\cong\mathrm{Alg}(K^H,k)$ is a group under convolution. More directly, if $f,f'\in\mathrm{Alg}_H(K,k)$ then

$$\begin{aligned} f*f'(xy) &= f\otimes f'(x_1(x_2)_{-1}y_1\otimes(x_2)_0y_2)=f(x_1)f'(y_1)f(x_2)f'(y_2)\\ &= (f*f')(x)(f*f')(y)\\ f*f'(hx) &= f\otimes f'(h_1x_1\otimes h_2x_2)=f(h_1x_1)f'(h_2x_2)=\varepsilon(h)(f*f')(x), \end{aligned}$$

and $f*fs=\varepsilon=fs*f$, so that $\mathrm{Alg}_H(K,k)$ is closed under convolution multiplication and inversion.

For $F \in {}_H\mathrm{Alg}_H(K\#H, k)$ the map $\mathrm{res}(F)\colon K \to k$ is in fact a H-invariant algebra map, since

$$\begin{aligned}\mathrm{res}(F)(hx) &= F(hx \otimes 1) = F((1 \otimes h_1)(x \otimes 1)(1 \otimes s(h_3)) \\ &= \varepsilon(h)F(x \otimes 1) = \varepsilon(h)\,\mathrm{res}(F)(x)\end{aligned}$$

and

$$\mathrm{res}(F)(xy) = F(xy \otimes 1) = F(x \otimes 1)F(y \otimes 1) = res(F)(x)\,\mathrm{res}(F)(y).$$

If $F' \in_H \mathrm{Alg}_H(K\#H, k)$ as well, then

$$\begin{aligned}\mathrm{res}(F * F')(x) &= F \otimes F'(x_1 \otimes (x_2)_{-1} \otimes (x_2)_0 \otimes 1) \\ &= F(x_1 \otimes 1)F'(x_2 \otimes 1) = \mathrm{res}(F) * \mathrm{res}(F')(x),\end{aligned}$$

showing that res is a group homomorphism. It is now easy to see that res is invertible and that the inverse is as stated.

As a composite of two group homomorphisms Θ is obviously a group homomorphism. Moreover,

$$\Theta(f)(1 \otimes h) = \mathrm{res}^{-1}(f) * 1 * \mathrm{res}^{-1}(f)s(1 \otimes h) = \varepsilon(h_1)(1 \otimes h_2)\varepsilon(h_3) = 1 \otimes h$$

for $f \in \mathrm{Alg}_H(K, k)$, showing that $\Theta(f)_{|H} = id$. □

Corollary 2.5. *Let R be a braided Hopf algebra in the braided category of crossed H-modules and let K be a braided Hopf subalgebra. If I is a Hopf ideal in K and $f \in \mathrm{Alg}_H(K, k)$ then,*

- *$J = I\#H$ and $J_f = \Theta(f)(J)$ are Hopf ideals in $K\#H$,*
- *$R\#H/(J) = R/(I)\#H$ and $R\#H/(J_f)$ are monoidally Morita-Takeuchi equivalent, if $(R\#H)/(\mathrm{res}^{-1}(f) * J) \neq 0$.*

Proposition 2.6. *Let K be a (braided) Hopf subalgebra of the (braided) Hopf algebra R such that R is left or right faithfully flat over K, and let $B = R/RK^+R$, where RK^+R is the Hopf ideal of R generated by K^+. Then:*

(1) *B is a (braided) Hopf algebra and*

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & R \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ k & \xrightarrow{\iota} & B \end{array}$$

is a pushout diagram of (braided) Hopf algebras.

(2) *$R^{coB} \cong K \cong {}^{coB}R$.*

(3) *The commutative square*

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & R \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ k & \xrightarrow{\iota} & B \end{array}$$

is a pullback of (braided) Hopf algebras.

Proof. Since $K^+ = \ker(\varepsilon)$ is a Hopf ideal in K it follows that RK^+R is a Hopf ideal in R and hence that $B = R/RK^+R$ is a (braided) Hopf algebra quotient of R.

First show that

$$K \xrightarrow{\kappa} R \overset{f}{\underset{g}{\rightrightarrows}} R \otimes_K R,$$

where $f = (1 \otimes \iota\varepsilon)\Delta$, $f(r) = r \otimes 1$, and $g = (\iota\varepsilon \otimes 1)\Delta$, $g(r) = r \otimes 1$, is exact, i.e: an equalizer diagram by faithful flatness ([Wa], Theorem 13.1). Observe that both f and g are injective maps. Consider the diagram

$$M \xrightarrow{w} M \otimes_K R \overset{u}{\underset{v}{\rightrightarrows}} M \otimes_K R \otimes_K R$$

with $w = 1 \otimes \iota$, $u = 1 \otimes f$ and $v = 1 \otimes g$. Then $w(m) = m \otimes 1$, $u(m \otimes r) = m \otimes r \otimes 1$, $v(m \otimes r) = m \otimes 1 \otimes r$ and $uw = vw$. If $N = \mathrm{eq}(u, v)$ then M is contained in N and the diagram

$$N \otimes_K R \to M \otimes_K R \otimes_K R \overset{u\otimes 1}{\underset{v\otimes 1}{\rightrightarrows}} M \otimes_K \otimes_K R \otimes_K R \otimes_K R$$

is exact by the flatness of R over K. Define $s\colon M \otimes_K \otimes_K R \otimes_K R \otimes_K R \to M \otimes_K R \otimes_K R$ by $s = 1 \otimes 1 \otimes m$, $s(m \otimes r \otimes r' \otimes r'') = m \otimes r \otimes r'r''$. If $x = \sum m_i \otimes r_i \otimes r'_i \in \mathrm{eq}(u \otimes 1, v \otimes 1) = N \otimes_K R$ then

$$\sum m_i \otimes r_i \otimes 1 \otimes r'_i = (u \otimes 1)x = (v \otimes 1)x = \sum m_i \otimes 1 \otimes r_i \otimes r'_i$$

and hence

$$x = s(u\otimes 1)x = s(v\otimes 1) = \sum m_i \otimes 1 \otimes r_i r'_i = (w\otimes 1)(\sum m_i \otimes r_i r'_i) \in \mathrm{im}(M \otimes_K R)$$

which shows that $M \otimes_K P = N \otimes_K R = \mathrm{eq}(u \otimes 1, v \otimes 1)$ and $N/M \otimes_K R \cong n \otimes_K R/M \otimes_K R = 0$. This implies that $N/M = 0$ by faithful flatness of R over K, and hence that $M = N$. In particular, if $M = K$ this gives

$$K = \mathrm{eq}(u, v) = \mathrm{eq}(f, g).$$

Now if $\pi\colon R \to B = R/RK^+R$ is the canonical projection then

$$R^{coB} \to R \overset{\phi}{\underset{\psi}{\rightrightarrows}} R \otimes B,$$

where $\phi = (1 \otimes \iota_B\varepsilon_R)\Delta_R$, $\phi(x) = x \otimes 1$, and $\psi = (1 \otimes \pi)\Delta_R$, $\psi(x) = x_1 \otimes \pi(x_2)$, is an exact equalizer diagram by definition. The Galois map $\beta = (m_R \otimes \pi)(1 \otimes \Delta_R)\colon R \otimes_K R \to R \otimes B$, defined by $\beta(x \otimes y) = xy_1 \otimes \pi(y_2)$, has inverse

$\beta^{-1}\colon R\otimes B \to R\otimes_K R$ given by $\beta^{-1}(1\otimes p) = (m_R\otimes 1)(1\otimes s\otimes 1)(1\otimes \Delta_R)$, $\beta^{-1}(x\otimes \pi(y)) = xs(y_1)\otimes y_2$. The diagram

$$\begin{tikzcd} K \arrow[r] & R \arrow[r, "f-g"] \arrow[d, equal] & R\otimes_K R \arrow[d, "\beta"'] \\ R^{coB} \arrow[r] & R \arrow[r, "\phi-\psi"] & R\otimes B \end{tikzcd}$$

commutes, so that $K \cong R^{coB}$. With the same argument, but on the left, one gets $K \cong {}^{coB}R$.

For any commutative diagram of (braided) Hopf algebras

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\varepsilon_X"'] & R \arrow[d, "\pi"'] \\ k \arrow[r, "\iota_B"] & B \end{tikzcd}$$

which is commutative, $\pi\alpha = \iota_B\varepsilon_X$, we get

$$\phi\alpha = (1\otimes \iota_B\varepsilon_R)\Delta_R\alpha = (\alpha\otimes\iota_B\varepsilon_X)\Delta_X = (1\otimes\pi)\Delta_R\alpha = \Psi\alpha.$$

Thus, since

$$K \to R \overset{\phi}{\underset{\psi}{\rightrightarrows}} R\otimes B$$

is an equalizer diagram, there is a unique Hopf algebra map $\gamma\colon X\to K$ such that $\kappa\gamma = \alpha$ and $\varepsilon_K\gamma = \varepsilon_X$, which shows that the diagram in item (3) is a pullback of (braided) Hopf algebras. $\square$

Observe that a Hopf algebra with cocommutative coradical is faithfully flat over any of its Hopf subalgebra [Ta].

2.5. **An exact 5-term sequence.** Let $ad_l\colon R\otimes R\to R$ and $ad_r\colon R\otimes R\to R$ be the left and the right adjoint actions, that is $ad_l(r\otimes r') = r_1a's(r_2)$ and $ad_r(r'\otimes r) = s(r_1)r'r_2$, respectively. A Hopf subalgebra K of the Hopf algebra R is said to be normal if it is stable under the left and the right adjoint action of R.

Lemma 2.7. *Let K be a Hopf subalgebra of R.*

(1) *If K is normal in R, then $RK^+ = K^+R$ is a Hopf ideal of R, $R/RK^+ \cong R\otimes_K k$ is a Hopf algebra, $\pi\colon R\to R/RK^+$ is Hopf algebra map and*

$$\begin{tikzcd} K \arrow[r, "\kappa"] \arrow[d, "\varepsilon"'] & R \arrow[d, "\pi"'] \\ k \arrow[r, "\iota"] & R/RK^+ \end{tikzcd}$$

is a pushout of Hopf algebras.

(2) *If $\pi R \to R'$ is a Hopf algebra map then*

$$R^{coR'} = \mathrm{eq}(\Delta(1 \otimes \iota\varepsilon), \Delta(1 \otimes \pi) \colon R \to R \otimes R'$$

is stable under the left and the right adjoint action and

$$\begin{array}{ccc} R^{coR'} & \xrightarrow{\kappa} & R \\ \varepsilon \Big\downarrow & & \pi \Big\downarrow \\ k & \xrightarrow{\iota} & R' \end{array}$$

is a pullback.

Proof. If $x \in R$ and $y \in K^+$ then $xy = ad_l(x_1 \otimes y)x_2$ and $yx = x_1 ad_r(y \otimes x_2)$. Since K is normal in R it follows that $RK^+ = K^+R$ and $I = RK^+$ is an ideal. It is a Hopf ideal, since RK^+ is always a coideal and since $s(I) = I$. Observe that

$$R \otimes K \underset{\varepsilon \otimes 1}{\overset{m(1 \otimes \kappa)}{\rightrightarrows}} R \xrightarrow{\pi} R/RK^+$$

is a coequalizer. If $u \colon R \to X$ is a Hopf algebra map such that $u\kappa = \iota\varepsilon$, then $um_R(1 \otimes \kappa) = m_X(u \otimes u)(\kappa \otimes 1) = m_X(\iota\varepsilon \otimes u)\varepsilon \otimes u$. Thus, there is a unique Hopf algebra map $u' \colon R/RK^+ \to X$ such that $u'\pi = u$ and $u'\iota = \iota_X$, showing that the diagram in (1) is a pushout.

If $y \in R^{coR'}$ then $ad_l(x \otimes y) = x_1 y s(x_2)$ and $(1 \otimes \pi)\Delta ad_l(x \otimes y) = x_1 y_1 s(x_4) \otimes \pi(x_2 y_2 s(x_3)) = x_1 y s(x_2) \otimes 1 = (1 \otimes \varepsilon)\Delta ad_l(x \otimes y)$ and similarly for $ad_r(y \otimes x)$. The diagram obviously commutes. If $v \colon Z \to A$ is a Hopf algebra map such that $\pi v = \iota_{R'}\varepsilon_Z$ then $(1 \otimes \pi)\Delta v = (v \otimes \pi v)\Delta = (v \otimes \iota_{R'}\varepsilon_Z)\Delta = (1 \otimes \iota\varepsilon)(v \otimes v)\Delta = (1 \otimes \iota\varepsilon)\Delta v$. Hence there is a unique Hopf algebra map $v' \colon Z \to R^{coR'}$ such that $\kappa v' = \varepsilon v'$, and the diagram in (2) is a pullback. □

Proposition 2.8. *Let K be a (braided) Hopf subalgebra of R such that R is left or right faithfully flat over K, and such that $RK^+ = K^+R$. If $B = R/RK^+$. Then:*

(1) $R^{coB} = K = {}^{coB}R$.
(2) *K is a normal Hopf subalgebra of R.*
(3) *There are spectral sequences*

$$H^p(B, H^q(K, Y)) \overset{p}{\Longrightarrow} H^n(R, Y),$$

$$H_p(B, H_q(K, X)) \overset{p}{\Longrightarrow} H_n(R, X).$$

(4) *Exact sequences in low degrees*

$$0 \to H^1(B, Y) \to H^1(R, Y) \to Hom_B(K^+ \otimes_K k, Y) \to H^2(B, Y) \to H^2(R, Y),$$

$$H_2(R, X) \to H_2(B, X) \to (K^+ \otimes_K k) \otimes_B X \to H_1(R, X) \to H_1(B, X) \to 0.$$

Proof. For items (1) 1nd (2) see [Mo], Proposition 3.4.3. The spectral sequences are special cases of those for normal subalgebras [CE], Chap. XVI, Theorem 6.1, where it actually suffices to assume that R is flat as a K-module. The natural isomorphisms $H^p(R,Y) \cong \mathrm{Ext}^p_R(k,Y)$ and $H_p(R,X) \cong \mathrm{Tor}^R_1(k,X)$, valid for any Hopf algebra R, have been used. Finally, the low degree 5-term sequences of these spectral sequences are those of [CE], page 329, cases C and C', where the isomorphisms

$$E_2^{01} = \mathrm{Hom}_B(k, H^1(K,Y)) \cong Hom_B(k, \mathrm{Hom}_K(K^+,Y)) \cong \mathrm{Hom}_B(K^+ \otimes_K k, Y)$$

and

$$E^2_{01} = H_1(K,k) \otimes_B X \cong (K^+ \otimes_K k) \otimes_B X$$

have been used. □

The 5-term exact sequences can also be found directly without the use of spectral sequences. The exact sequence of K-modules $0 \to K^+ \to K \to k \to 0$ induces an exact sequence of R-modules $0 \to K^+ \otimes_K R \to K \otimes_K R \to k \otimes_K R \to 0$, that is $K^+ \otimes_K R = K^+R$ and $k \otimes_K R = R/K^+R$, since R is K-flat. Again, since R is K-flat, any R-projective (or R-flat) resolution $\mathbf{X}$ of an R-module M is also a K-flat resolution of M. For every injective K-module map $f\colon Y \to Y'$ the R-module map $1 \otimes f : R \otimes_K Y \to R \otimes_K Y'$ is injective, since R is K-flat, and thus $f \otimes_K 1 : Y \otimes_K X \cong (Y \otimes_R R) \otimes_K X \to (Y' \otimes_R R) \otimes_K X \cong Y' \otimes_K X$ is injective as well for every flat R-module X. This gives an isomorphism of complexes

$$k \otimes_K \mathbf{X} \cong k \otimes_K R \otimes_R \mathbf{X} \cong B \otimes_R \mathbf{X},$$

so that $H_n(K,M) = \mathrm{Tor}^K_n(k,M) \cong \mathrm{Tor}^R_n(B,M)$. From the exact sequence of R-modules $0 \to R^+ \to R \to k \to 0$ we then get a commutative diagram of B-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}^R_1(B,k) \arrow[r] \arrow[d, "\cong"'] & B \otimes_R R^+ \arrow[r] \arrow[d, equal] & B \otimes_R R \arrow[r] \arrow[d, "\cong"'] & B \otimes_R k \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & H_1(K,k) \arrow[r] & B \otimes_R R^+ \arrow[r] & B \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

and hence an exact sequence of B-modules

$$0 \to K^+/(K^+)^2 \to B \otimes_R R^+ \to B^+ \to 0.$$

Apply the functor $\mathrm{Hom}_B(\ ,M)$ to this last exact sequence to get

$$\begin{aligned} 0 \quad \to \quad & \mathrm{Hom}_B(B^+,M) \to \mathrm{Hom}_B(B \otimes_R R^+, M) \to \mathrm{Hom}_B(H_1(K,k),M) \\ \to \quad & \mathrm{Ext}^1_B(B^+,M) \to \mathrm{Ext}^1_B(B \otimes_R R^+, M) \end{aligned}$$

If the bottom sequence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & P \arrow[r, "\tilde{p}"] \arrow[d, "\beta"'] & R^+ \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & M \arrow[r] & Y \arrow[r, "p"] & B \otimes_R R^+ \arrow[r] & 0
\end{tikzcd}$$

is an extension E of B-modules then it is also an extension of R-modules. The map $\alpha\colon R^+ \to B\otimes_R R^+$, given by $\alpha(x) = 1\otimes_R x$, is an R-module homomorphism. The right hand square of the diagram is a pullback and the top sequence is a representative of $\alpha^*(E)$. If the top sequence is split by $\tilde{u}\colon R^+ \to P$ then $u\colon B\otimes_R R^+ \to Y$, given by $u(b\otimes x) = b\beta\tilde{u}(x)$, splits the bottom sequence, since $pu(b\otimes x) = bp\beta u(x) = b\alpha\tilde{p}u(x) = b\alpha(x) = b\otimes x$. Hence the induced map $\alpha^*\colon \mathrm{Ext}^1_B(B\otimes_R R^+, M) \to \mathrm{Ext}^1_R(R^+, M)$ is injective. Moreover, $\mathrm{Ext}^1_R(R^+, M) \cong H^2(R, M)$.

Using $\mathrm{Hom}_B(B\otimes_R R^+, M) \cong \mathrm{Hom}_R(R^+, M)$ one gets the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & \mathrm{Hom}_B(B,M) \arrow[r, equal] \arrow[d, "u^*"'] & \mathrm{Hom}_R(R,M) \arrow[d, "v^*"'] & \\
0 \arrow[r] & \mathrm{Hom}_B(B^+,M) \arrow[r, "\kappa^*"] \arrow[d] & \mathrm{Hom}_R(R^+,M) \arrow[r, "\delta"] \arrow[d, "\partial"'] & \mathrm{Hom}_B(H_1(K,k),M) \\
0 \arrow[r] & \mathrm{Ext}^1_B(k,M) \arrow[r] \arrow[d] & \mathrm{Ext}^1_R(k,M) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

in which $\delta v^* = \delta\kappa^* u^* = 0$, so that there is a unique homomorphism

$$\gamma\colon\ \mathrm{Ext}^1_R(k, M) \to \mathrm{Hom}_B(H_1(K,k), M)$$

such that $\gamma\partial = \delta$. It follows that the sequence

$$0 \to H^1(B,M) \to H^1(R,M) \xrightarrow{\gamma} \mathrm{Hom}_B(H_1(K,k),M) \to H^2(B,M) \to H^2(R,M)$$

is exact. Similar argument work for the homology sequence.

3. Liftings over finite abelian groups

In this section we give a somewhat different characterization of the class of finite dimensional pointed Hopf algebras classified in [AS], and show that any two such Hopf algebras with isomorphic associated graded Hopf algebras are monoidally Morita-Takeuchi equivalent, and therefore cocycle deformations of each other, as we will point out in the next section.

A datum of finite Cartan type

$$\mathcal{D} = \mathcal{D}\left(G, (g_i)_{1\le i\le\theta}, (a_{ij})_{1\le i,j\le\Theta}\right)$$

for a (finite) abelian group G consists of elements $g_i \in G$, $\chi_j \in \widehat{G}$ and a Cartan matrix (a_{ij}) of finite type satisfying the Cartan condition

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}$$

with $q_{ii} \neq 1$, where $q_{ij} = \chi_j(g_i)$, in particular $q_{ii}^{a_{ij}} = q_{jj}^{a_{ji}}$ for all $1 \leq i, j \leq \theta$. In general, the matrix (q_{ij} of a diagram of Cartan type is not symmetric, but by [AS, Lemma 1.2] it can be reduced to the symmetric case by twisting.

Let $\mathbf{Z}[I]$ be the free abelian group of rank θ with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_\theta\}$. The Weyl group $W \subset \mathrm{Aut}(\mathbf{Z}[I])$ of (a_{ij}) is generated by the reflections $s_i \colon \mathbf{Z}[I] \to \mathbf{Z}[I]$, where $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all i, j. The root system of the Cartan matrix (a_{ij}) is $\Phi = \cup_{i=1}^{\theta} W(\alpha_i)$ and $\Phi^+ = \Phi \cap [I] = \left\{ \alpha \in \Phi |\, \alpha = \sum_{i=1}^{\theta} n_i\alpha_i, n_i \geq 0 \right\}$ is the set of positive roots relative to the basis of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_\theta\}$. Obviously, the number of positive roots $p = |\Phi^+|$ is at least θ. The maps $g \colon \mathbf{Z}[I] \to G$ and $\chi \colon \mathbf{Z}[I] \to \tilde{G}$ given by $g_\alpha = g_1^{n_1} g_2^{n_2} \ldots g_\theta^{n_\theta}$ and $\chi_\alpha = \chi^{n_1}\chi_2^{n_2} \ldots \chi_\theta^{n_\theta}$ for $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i$, respectively, are group homomorphisms. The bilinear map $q \colon \mathbf{Z}[I] \times \mathbf{Z}[I] \to k^x$ defined by $q_{\alpha_i\alpha_j} = q_{ij}$ can be expressed as $q_{\alpha\beta} = \chi_\beta(g_\alpha)$.

If $\mathcal{X}$ the set of connected components of the Dynkin diagram of Φ let Φ_J be the root system of the component $J \in \mathcal{X}$. The partition of the Dynkin diagram into connected components corresponds to an equivalence relation on $I = \{1, 2, \ldots, \theta\}$, where $i \sim j$ if α_i and α_j are in the same connected component.

Lemma 3.1. [AS, Lemma 2.3] *Suppose that $\mathcal{D}$ is a connected datum of finite Cartan type, i.e: the Dynkin diagram of the Cartan matrix (a_{ij}) is connected, and such that*

(1) *q_{ii} has odd order, and*
(2) *the order of q_{ii} is prime to 3, if (a_{ij}) is of type G_2.*

Then there are integers $d_i \in \{1, 2, 3\}$ for $1 \leq i \leq \theta$ and a $q \in k^x$ of odd order N such that

$$q_{ii} = q^{2d_i} \quad . \quad d_i a_{ij} = d_j a_{ji}$$

for $1 \leq i, j \leq \Theta$. If the Cartan matrix (a_{ij}) of $\mathcal{D}$ is of type G_2 then the order of q is prime to 3. In particular, the q_{ii} all have the same order in $k^\times$, namely N.

More generally, let $\mathcal{D}$ be a datum of finite Cartan type in which the order N_i of q_{ii} is odd for all i, and the order of q_{ii} is prime to 3 for all i in a connected component of type G_2. It then follows that the order function N_i is constant, say equal to N_J, on each connected component J. A datum satisfying these conditions will be called special datum of finite Cartan type.

Fix a reduced decomposition of the longest element

$$w_0 = s_{i_1} s_{i_2} \ldots s_{i_p}$$

of the Weyl group W in terms of the simple reflections. Then

$$\left\{ s_{i_1} s_{i_2} \ldots s_{i_{l-1}}(\alpha_{i_l}) \right\}_{i=1}^{p}$$

is a convex ordering of the positive roots.

Let $V = V(\mathcal{D})$ be the crossed kG-module with basis $\{x_1, x_2, \ldots, x_\theta\}$, where $x_i \in V_{g_i}^{\chi_i}$ for $1 \leq i \leq \theta$. Then for all $1 \leq i \neq j \leq \Theta$ the elements $ad^{1-a_{ij}} x_i(x_j)$ are primitive in the free braided Hopf algebra $\mathcal{A}(V)$ (see Lemma 1.6 or [AS1,

Appendix 1]). If $\mathcal{D}$ is as in the previous Lemma then $\chi^{1-a_{ij}}\chi_j \neq \varepsilon$. This implies that $f(u_{ij}) = 0$ for any braided (Hopf) subalgebra A of $\mathcal{A}(V)$ containing $u_{ij} = ad^{1-a_{ij}}x_i(x_j)$ and any G-invariant algebra map $f\colon A \to k$. Define root vectors in $\mathcal{A}(V)$ as follows by iterated braided commutators of the elements $x_1, x_2, \dots, x_\theta$, as in Lusztig's case but with the general braiding:

$$x_{\beta_l} = T_{i_1}T_{i_2}\dots T_{i_{l-1}}(x_{i_l}),$$

where $T_i(x_j) = \mathrm{ad}_{x_i}^{-a_{ij}}(x_j)$

In the quotient Hopf algebra $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}}x_i(x_j)|1 \leq i \neq j \leq \theta)$ define root vectors $x_\alpha \in \mathcal{A}(V)$ for $\alpha \in \Phi^+$ by the same iterated braided commutators of the elements $x_1, x_2, \dots, x_\theta$ as in Lusztig's case but with respect to the general braiding. (See [AS2], and the inductive definition of root vectors in [Ri] or also [CP, Section 8.1 and Appendix].) Let $K(\mathcal{D})$ be the subalgebra of $R(\mathcal{D})$ generated by $\left\{ x_\alpha^N \middle| \alpha \in \Phi^+ \right\}$.

Theorem 3.2. [AS, Theorem 2.6] *Let $\mathcal{D}$ be a connected datum of finite Cartan type as in the previous Lemma. Then*

(1) $\left\{ x_{\beta_1}^{a_1}x_{\beta_2}^{a_2}\dots x_{\beta_p}^{a_p} \middle| a_1, a_2, \dots, a_p \geq 0 \right\}$ *forms a basis of* $R(\mathcal{D})$,

(2) $K(\mathcal{D})$ *is a braided Hopf subalgebra of* $R(\mathcal{D})$ *with basis*

$$\left\{ x_{\beta_1}^{Na_1}x_{\beta_2}^{Na_2}\dots x_{\beta_p}^{Na_p} \middle| a_1, a_2, \dots, a_p \geq 0 \right\},$$

(3) $[x_\alpha, x_\beta^N]_c = 0$, *i.e:* $x_\alpha x_\beta^N = q_{\alpha\beta}^N x_\beta^N x_\alpha$ *for all* $\alpha, \beta \in \Phi^+$.

The vector space $V = V(\mathcal{D})$ can also be viewed as a crossed module in ${}^{\mathbf{Z}[I]}_{\mathbf{Z}[I]}YD$. The Hopf algebra $\mathcal{A}(V)$, the quotient Hopf algebra $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}}x_i(x_j)|1 \leq i \neq j \leq \theta)$ and its Hopf subalgebra $K(\mathcal{D})$ generated by $\left\{ x_\alpha^N \middle| \alpha \in \Phi^+ \right\}$ are all Hopf algebras in ${}^{\mathbf{Z}[I]}_{\mathbf{Z}[I]}YD$. In particular, their comultiplications are $\mathbf{Z}[I]$-graded. By construction, for $\alpha \in \Phi^+$, the root vector $x_\alpha \in R(\mathcal{D})$ is $\mathbf{Z}[I]$-homogeneous of $\mathbf{Z}[I]$-degree α, so that $x_\alpha \in R(\mathcal{D})_{g_\alpha}^{\chi_\alpha}$. To simplify notation write for $1 \leq l \leq p$ and for $\underline{a} = (a_1, a_2, \dots, a_p) \in \mathbf{N}^p$

$$h_l = g_{\beta_l}^N\ ,\ \eta_l = \chi_{\beta_l}^N\ ,\ z_l = x_{\beta_l}^N$$

and $\underline{a} = \sum_{i=1}^p a_i\beta_i$

$$h^{\underline{a}} = h_1^{a_1}h_2^{a_2}\dots h_p^{a_p} \in G\ ,\ \eta^{\underline{a}} = \eta_1^{a_1}\eta_2^{a_2}\dots\eta_p^{a_p} \in \tilde{G}\ ,\ z^{\underline{a}} = z_1^{a_1}z_2^{a_2}\dots z_p^{a_p} \in K(\mathcal{D}).$$

In particular, for $e_l = (\delta_{kl})_{1 \leq k \leq l}$, where δ_{kl} is the Kronecker symbol, $e_l = \beta_l$ and $z^{e_l} = z_l$ for $1 \leq l \leq p$. The height of $\alpha = \sum_{i=1}^\theta n_i\alpha_i \in \mathbf{Z}[I]$ is defined to be the integer $ht(\alpha) = \sum_{i=1}^\theta n_i$. Observe that if $\underline{a}, \underline{b}, \underline{c} \in \mathbf{N}^p$ and $\underline{a} = \underline{b} + \underline{c}$ then

$$h^{\underline{a}} = h^{\underline{b}}h^{\underline{c}}\ ,\ \eta^{\underline{a}} = \eta^{\underline{b}}\eta^{\underline{c}} \text{ and } \mathrm{ht}(\underline{b}) < \mathrm{ht}(\underline{a}) \text{ if } \underline{c} \neq 0.$$

The comultiplication on $K(\mathcal{D})$ is $\mathbf{Z}[I]$-graded, so that

$$\Delta_{K(\mathcal{D})}(z^{\underline{a}}) = z^{\underline{a}} \otimes 1 + 1 \otimes z^{\underline{a}} + \sum_{\underline{b},\underline{c}\neq 0;\underline{b}+\underline{c}=\underline{a}} t^{\underline{a}}_{\underline{bc}} z^{\underline{b}} \otimes z^{\underline{c}}$$

and hence

$$\Delta_{K(\mathcal{D})\#kG}(z^{\underline{a}}) = z^{\underline{a}} \otimes 1 + h^{\underline{a}} \otimes z^{\underline{a}} + \sum_{\underline{b},\underline{c}\neq 0;\underline{b}+\underline{c}=\underline{a}} t^{\underline{a}}_{\underline{bc}} z^{\underline{b}} h^{\underline{c}} \otimes z^{\underline{c}}$$

on the bosonization. The algebra $K(\mathcal{D})$ is generated by the subspace $L(\mathcal{D})$ with basis $\{z_1, z_2, \dots, z_p\}$. The (left) kG-module structure on $\mathcal{A}(V)$ restricts to $L(\mathcal{D})$, and induces (right) kG-actions on $\mathrm{Alg}(K(\mathcal{D}), k)$ and on $\mathrm{Vect}(L(\mathcal{D}), k)$ by the formula $(fg)(x) = f(gx)$. A linear functional $f\colon L(\mathcal{D}) \to k$ is called g-invariant if $fg = f$ for all $g \in G$. Let $\mathrm{Vect}_G(L(\mathcal{D}), k)$ be the subspace of G-invariant linear functionals in $\mathrm{Vect}(L(\mathcal{D}), k)$.

Proposition 3.3. *Let $\mathrm{Vect}_G(L(\mathcal{D})$ and $\mathrm{Alg}_G(K(\mathcal{D}), k)$ be the space of G-invariant linear functionals and the set of G-invariant algebra maps, where $\mathcal{D}$ is a connected special datum of finite Cartan type. Then:*

(1) $\mathrm{Vect}_G(L(\mathcal{D}), k) = \{ f \in \mathrm{Vect}(L(\mathcal{D}), k) | \, f(z_l) = 0 \text{ if } \eta_l \neq \varepsilon\}$.
(2) *The restriction map* $\mathrm{res}\colon \mathrm{Alg}_G(K(\mathcal{D}), k) \to \mathrm{Vect}_G(L(\mathcal{D}), k)$ *is a bijection. The inverse is given by* $\mathrm{res}^{-1}(f)(z^{\underline{a}}) = f(z_1)^{a_1}) f(z_2^{a_2}) \dots f(z_p^{a_p})$.
(3) $\mathrm{Alg}_G(K(\mathcal{D}), k)$ *is a group under convolution.*
(4) *The restriction map* $\mathrm{res}\colon {}_G\mathrm{Alg}_G(K(\mathcal{D})\#kG, k) \to \mathrm{Alg}_G(K(\mathcal{D}), k)$ *is an isomorphism of groups with inverse defined by* $\mathrm{res}^{-1}(f)(x \otimes g) = f(x)$, *and* ${}_G\,\mathrm{Alg}_G(K(\mathcal{D})\#kG, k) = \left\{ \tilde{f} \in \mathrm{Alg}(K(\mathcal{D})\#kG, k) \middle| \, \tilde{f}_{|kG} = \varepsilon \right\}$.
(5) *The map* $\Theta = \rho\,\mathrm{res}^{-1}\colon \mathrm{Alg}_G(K(\mathcal{D}), k) \to \mathrm{Aut}_{Hopf}(K(\mathcal{D})\#kG)$, *defined by* $\Theta(f) = \mathrm{res}^{-1}(f) * 1 * \mathrm{res}^{-1}(f)s$, *is a group homomorphism whose image is a subgroup in*
$$\widetilde{\mathrm{Aut}}_{Hopf}(K(\mathcal{D})\#kG) = \left\{ f \in \mathrm{Aut}_{Hopf}(K(\mathcal{D})\#kG) | \, f_{|kG} = id \right\}.$$
(6) *For every $f \in \mathrm{Alg}_G(K(\mathcal{D}), k)$ the automorphism $\Theta(f)$ of $K(\mathcal{D})\#kG$ is determined by*
$$\Theta(f) z^{\underline{a}} = z^{\underline{a}} + f(z^{\underline{a}})(1 - h^{\underline{a}}) + \sum_{\underline{b},\underline{c}\neq 0;\underline{b}+\underline{c}=\underline{a}} t^{\underline{a}}_{\underline{bc}} f(z^{\underline{b}}) z^{\underline{c}}$$
$$+ \sum_{\underline{b},\underline{c}\neq 0;\underline{b}+\underline{c}=\underline{a}} t^{\underline{a}}_{\underline{bc}} \left[z^{\underline{b}} + f(z^{\underline{b}})(1 - h^{\underline{b}}) + \sum_{\underline{d},\underline{e}\neq 0;\underline{d}+\underline{e}=\underline{b}} t^{\underline{b}}_{\underline{d},\underline{e}} f(z^{\underline{d}}) z^{\underline{e}} \right] h^{\underline{c}} f s(z^{\underline{c}}).$$
In particular, $\Theta(z^{\underline{a}}) = z^{\underline{a}} + f(z^{\underline{a}})(1 - h^{\underline{a}})$ *if* $ht(\underline{a}) = 1$.

Proof. If $f \in \mathrm{Vect}_G(L(\mathcal{D}), k)$ then $f(z_i) = f(gz_i) = \eta_i(g) f(z_i)$ for all $1 \leq i \leq p$ and for all $g \in G$. Thus, $f(z_i) = 0$ if $\eta_i \neq \varepsilon$.

By Theorem 3.2 it follows that $K(\mathcal{D}) \cong TL(\mathcal{D})/(z_iz_j - \eta_j(h_i)z_jz_i|1 \leq i < j \leq p)$. If $f \in \mathrm{Vect}_G(L(\mathcal{D}), k)$ then the induced algebra map $\tilde{f}\colon TL(\mathcal{D}) \to k$ factors uniquely through $K(\mathcal{D})$, since $\tilde{f}(z_iz_j - \eta_j(h_i)z_jz_i) = f(z_i)f(z_j) - \eta_j(h_i)f(z_j)f(z_i) = f(z_i)(f(z_j) - f(h_iz_j)) = 0$ for $1 \leq i, j \leq p$, by the fact that f is G-invariant. This proves the second assertion.

The next three assertions are a special case of 2.4.

The set of all algebra maps $\mathrm{Alg}(K(\mathcal{D}), k)$ may not be a group under convolution, but the subset $\mathrm{Alg}_G(K(\mathcal{D}), k)$ is. If f_1, f_2 and f are G-invariant then

$$\begin{aligned} f_1 * f_2(xy) &= (f_1 \otimes f_2)(m \otimes m)(1 \otimes c \otimes 1)(x_1 \otimes x_2 \otimes y_1 \otimes y_2) \\ &= (f_1 \otimes f_2)(x_1(x_2)_{-1}y_1 \otimes (x_2)_0 y_2) \\ &= f_1(x_1)\varepsilon((x_2)_{-1})f_1(y_1)f_2((x_2)_0)f_2(y_2) \\ &= f_1(x_1)f_1(y_1)f_2(x_2)f_2(y_2) = f_1 * f_2(x)f_1 * f_2(y) \end{aligned}$$

and moreover, $(f_1 * f_2)g = f_1g * f_2g = f_1 * f_2$, $fsg = fgs = fs$, $\varepsilon * f = f = f * \varepsilon$ and $f * fs = \varepsilon = fs * f$ so that $\mathrm{Alg}_G(K(\mathcal{D})$ is closed under convolution multiplication and inversion.

The map $\Psi\colon\ \mathrm{Alg}_G(k(\mathcal{D}), k) \to \mathrm{Alg}(K(\mathcal{D})\#kG, k)$ given by $\Psi(f)(x \otimes g) = f(x)$, is a homomorphism, since

$$\begin{aligned} \mathrm{res}^{-1}(f_1) * \mathrm{res}^{-1}(f_2)(x \otimes g) &= \mathrm{res}^{-1}(f_1) \otimes \mathrm{res}^{-1}(f_2)(x_1 \otimes (x_2)_{-1}g \otimes (x_2)_0 \otimes g) \\ &= f_1(x_1)\varepsilon((x_2)_{-1})f_2((x_2)_0) \\ &= f_1(x_1)f_2(x_2) = f_1 * f_2(x) \\ &= \mathrm{res}^{-1}(f_1) * \mathrm{res}^{-1}(f_2)(x \otimes g). \end{aligned}$$

The inverse $\Psi^{-1}\colon \widetilde{\mathrm{Alg}}(K(\mathcal{D})\#kG, k) \to \mathrm{Alg}_G(K(\mathcal{D}), k)$ is given by $\Psi^{-1}(\tilde{f})(x) = \tilde{f}(x \otimes 1)$, is just the restriction map.

It is convenient to use the notation $\Psi(f) = \tilde{f}$. Then $\Theta(f) = \tilde{f} * 1 * \tilde{f}s$ and

$$\begin{aligned} \Theta(f_1 * f_2) &= \widetilde{f_1 * f_2} * 1 * \widetilde{f_1 * f_2}s \\ &= \tilde{f}_1 * \tilde{f}_2 * 1 * \tilde{f}_2 s * \tilde{f}_1 s \\ &= \Theta(f_1)\Theta(f_2). \end{aligned}$$

In particular, $\Theta(f)\Theta(fs) = \Theta(f * fs) = \Theta(\varepsilon) = 1 = \Theta(fs * f) = \Theta(fs)\Theta(f)$. Moreover,

$$\begin{aligned} \Theta(f)(xy) &= \tilde{f}(x_1y_1)x_2y_2\tilde{f}s(x_3y_3) \\ &= \tilde{f}(x_1)\tilde{f}(y_1)x_2y_2\tilde{f}s(y_3)\tilde{f}(x_3) \\ &= \tilde{f}(x_1)x_2\tilde{f}s(x_3)\tilde{f}(y_1)y_2\tilde{f}s(y_3) \\ &= \Theta(f)(x)\Theta(f)(y) \end{aligned}$$

and

$$\begin{aligned}
\Delta\Theta(f) &= \Delta(\tilde{f} * 1 * \tilde{f}s) \\
&= \Delta(\tilde{f} \otimes 1 \otimes \tilde{f}s)\Delta^{(2)} \\
&= (\tilde{f} \otimes 1 \otimes 1\tilde{f}s)\Delta^{(3)} \\
&= (\tilde{f} \otimes 1 \otimes \varepsilon \otimes 1 \otimes \tilde{f}s)\Delta^{(4)} \\
&= (\tilde{f} \otimes 1 \otimes \tilde{f}s \otimes \tilde{f} \otimes 1\tilde{f}s)\Delta^{(5)} \\
&= (\tilde{f} * 1 * \tilde{f}s \otimes \tilde{f} * 1 * \tilde{f}s)\Delta \\
&= (\Theta(f) \otimes \Theta(f))\Delta,
\end{aligned}$$

showing that $\Theta(f)$ is an automorphism of $K(\mathcal{D})\#kG$ with inverse $\Theta(fs)$.

The remaining item now follows from the formula for the comultiplication

$$\Delta(z^{\underline{a}}) = z^{\underline{a}} \otimes 1 + h^{\underline{a}} \otimes z^{\underline{a}} + \sum_{\underline{b},\underline{c}\neq 0;\underline{b}+\underline{c}=\underline{a}} t^{\underline{a}}_{\underline{bc}} z^{\underline{b}} h^{\underline{c}} \otimes z^{\underline{c}}$$

of $K(\mathcal{D})\#kG$, which implies

$$\begin{aligned}
\Delta^{(2)}(z^{\underline{a}}) &= z^{\underline{a}} \otimes 1 \otimes 1 + h^{\underline{a}} \otimes z^{\underline{a}} \otimes 1 + \sum t^{\underline{a}}_{\underline{bc}} z^{\underline{b}} h^{\underline{c}} \otimes z^{\underline{c}} \otimes 1 + h^{\underline{a}} \otimes h^{\underline{a}} \otimes z^{\underline{a}} \\
&+ \sum t^{\underline{a}}_{\underline{bc}} \left[z^{\underline{b}} h^{\underline{c}} \otimes h^{\underline{c}} \otimes z^{\underline{c}} + h^{\underline{a}} \otimes z^{\underline{b}} h^{\underline{c}} \otimes z^{\underline{c}} + \sum t^{\underline{b}}_{\underline{rl}} z^{\underline{r}} h^{\underline{l}+\underline{c}} \otimes z^{\underline{l}} h^{\underline{c}} \otimes z^{\underline{c}} \right].
\end{aligned}$$

and

$$1 * s(z^{\underline{a}}) = z^{\underline{a}} + h^{\underline{a}} s(z^{\underline{a}}) + \sum t^{\underline{a}}_{\underline{bc}} z^{\underline{b}} h^{\underline{c}} s(z^{\underline{c}}) = \varepsilon(z^{\underline{a}}) = 0.$$

Applying $\tilde{f}$ to the latter gives

$$f(z^{\underline{a}}) + fs(z^{\underline{a}}) + \sum t^{\underline{a}}_{\underline{bc}} f(z^{\underline{b}}) fs(z^{\underline{c}}) = 0,$$

which will be used in the following evaluation. Now compute

$$\begin{aligned}
\Theta(f)(z^{\underline{a}}) &= f(z^{\underline{a}}) + z^{\underline{a}} + \sum t^{\underline{a}}_{\underline{bc}} f(z^{\underline{b}}) z^{\underline{c}} + h^{\underline{a}} fs(z^{\underline{a}}) \\
&\quad + \sum t^{\underline{a}}_{\underline{bc}} [f(z^{\underline{b}}) + z^{\underline{b}} + \sum t^{\underline{b}}_{\underline{rl}} f(z^{\underline{r}}) z^{\underline{l}}] h^{\underline{c}} fs(z^{\underline{c}}) \\
&= z^{\underline{a}} + f(z^{\underline{a}})(1 - h^{\underline{a}}) + \sum t^{\underline{a}}_{\underline{bc}} f(z^{\underline{b}}) z^{\underline{c}} \\
&\quad + \sum t^{\underline{a}}_{\underline{bc}} [z^{\underline{b}} + f(z^{\underline{b}})(1 - h^{\underline{b}}) + \sum t^{\underline{b}}_{\underline{rl}} f(z^{\underline{r}}) z^{\underline{l}}] h^{\underline{c}} fs(z^{\underline{c}})
\end{aligned}$$

to get the required result. □

For any $f \in \mathrm{Alg}_G(K(\mathcal{D}), k)$ define by induction on $ht(a)$ the following elements in the augmentation ideal of kG

$$u_{\underline{a}}(f) = f(z^{\underline{a}})(1 - h^{\underline{a}}) + \sum_{\underline{b},\underline{c}\neq 0;\underline{b}+\underline{c}=\underline{a}} t^{\underline{a}}_{\underline{bc}} f(z^{\underline{b}}) u_{\underline{c}}(f),$$

where $u_{\underline{a}}(f) = f(z^{\underline{a}})(1 - h^{\underline{a}})$ if $ht(\underline{a}) = 1$. In particular, for a positive root $\alpha = \beta_l \in \Phi^+$ and $x_\alpha^N = x_{\beta_l}^N = z^{e_l}$ write $u_l(f) = u_{e_l}(f) = u_\alpha(f)$. We can think of $f = (f(x_\alpha^N)|\alpha \in \Phi^+)$ as root vector parameters in the sense of [AS].

Corollary 3.4. *Let $\mathcal{D}$ be a special connected datum of Cartan type. Then*

$$U(\mathcal{D}, f) = R(\mathcal{D})\#kG/(x_\alpha^N + u_\alpha(f))$$

are the liftings of $\mathcal{B}(V)\#kG = U(\mathcal{D}, \varepsilon)$.

Proof. The augmentation ideal of $K(\mathcal{D})$, the ideal I of $K(\mathcal{D})\#kG$ and the ideal (I) in $R(\mathcal{D})\#kG$ generated by $\{x_\alpha |\, \alpha \in \mathcal{X}\}$ are Hopf ideals. It follows from the inductive formulas for $\Theta(f)(z^{\underline{a}})$ and $u_{\underline{a}}(f)$ above that for every $f \in \mathrm{Alg}_G(K(\mathcal{D}), k)$ the ideals $I_f = \Theta(f)(I)$ in $K(\mathcal{D})\#kG$ and (I_f) in $R(\mathcal{D})\#kG$ generated by $\left\{x_\alpha^N + u_\alpha(f)\middle|\, \alpha \in \Phi^+\right\}$ are Hopf ideals as well. The Hopf algebras $U(\mathcal{D}, f) = K(\mathcal{D})\#kG/(\Theta(f)(I))$ are the liftings of $U(\mathcal{D}, \varepsilon) = \mathcal{B}(V)\#kG$ parameterized by $f = (f(x_\alpha^N|\alpha \in \Phi^+) \in \mathrm{Alg}_G(K(\mathcal{D}), k)$. □

In the not necessarily connected case of a special datum of finite Cartan type the elements $ad^{1-a_{ij}}x_i(x_j)$ are still primitive in $\mathcal{A}(V)$ and $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-aij}x_i(x_j)|i \sim j)$ is still a Hopf algebra, which contains $R(\mathcal{D}_J)$ for every connected component $J \in \mathcal{X}$. The Hopf subalgebra $K(\mathcal{D})$ generated by the subspace with basis $S = \{z_J^{a_J}, z_{ij}|\, J \in \mathcal{X}, i \not\sim j\}$, where $z_{ij} = [x_i, x_j]_c$, contains $K(\mathcal{D}_J)$ for every $J \in \mathcal{X}$. The comultiplication in each components $K(\mathcal{D}_J)$ and $K(\mathcal{D}_J)\#kG$ is of course given as before in the connected case, while for $i \not\sim j$

$$\Delta(z_{ij}) = z_{ij} \otimes 1 + 1 \otimes z_{ij}$$

in $K(\mathcal{D})$ and $R(\mathcal{D})$ and

$$\Delta(z_{ij}) = z_{ij} \otimes 1 + g_i g_j \otimes z_{ij}$$

in the bozonizations $K(\mathcal{D})\#kG$ and $R(\mathcal{D})\#kG$. The space of G-invariant linear functionals $\mathrm{Vect}_G(L(\mathcal{D}), k)$ consists elements $f \in \mathrm{Vect}(L(\mathcal{D}), k)$ such that

$$f(z_r) = 0 \text{ if } \eta_\mathrm{r} \neq \varepsilon \text{ for } 1 \leq \mathrm{r} \leq \mathrm{p} \text{ and } \mathrm{f}(\mathrm{z_{ij}}) = 0 \text{ if } \chi_\mathrm{i}\chi_\mathrm{j} \neq \varepsilon \text{ if } \mathrm{i} \not\sim \mathrm{j}\}.$$

The induced algebra map $\tilde{f}\colon TL(\mathcal{D}) \to k$ of such a linear functional satisfies

- $\tilde{f}([z_r, z_s]_c) = f(z_r)(f(z_s) - f(h_r z_s)) = 0,$
- $\tilde{f}([z_{ij}, z_r]_c) = f(z_{ij})(f(z_r) - f(g_i g_j z_r) = 0,$
- $\tilde{f}([z_{ij}, z_{lm}]_c) = f(z_{ij})(f(z_{lm}) - f(g_i g_j z_{lm})) = 0,$

since f is G-invariant. It therefore factors through $K(\mathcal{D})$, since

$$TL(\mathcal{D})/([z_r, z_s]_c, [z_{ij}, z_r]_c, [z_{ij}, z_{lm}]_c) = K(\mathcal{D})/([z_r, z_s]_c, [z_{ij}, z_r]_c, [z_{ij}, z_{lm}]_c).$$

It follows that the restriction maps

$$\mathrm{res}\colon {}_G\mathrm{Alg}_G(K(\mathcal{D})\#kG) \to \mathrm{Alg}_G(K(\mathcal{D}), k) \to \mathrm{Vect}_G(L(\mathcal{D}), k)$$

are bijective, and $f = \{\, f(z_{ij}) |\, i \not\sim j\} \cup \{\, f(z_r) |\, 1 \le r \le p\}$ can be interpreted as a combination of linking parameters and root vector parameters in the sense of [AS]. Then map

$$\Theta : \mathrm{Alg}_G(K(\mathcal{D}), k) \to \mathrm{Aut}_{Hopf}(K(\mathcal{D})\#kG)$$

given by $\Theta(f) = f * 1 * fs$ is a homomorphism of groups. Moreover, since $z_{ij} + g_i g_j s(z_{ij}) = m(1 \otimes s)\Delta(z_{ij}) = 0$ in $K(\mathcal{D})\#kG$, it follows that

$$\Theta(f)(z_{ij}) = (f \otimes 1 \otimes fs)\Delta^{(2)}(z_{ij}) = z_{ij} + f(z_{ij})(1 - g_j g_j)$$

when $i \not\sim j$, while $\Theta f(z_r)$ is given inductively as in 3.3. In this way one obtains therefore all the 'liftings' of $B(V)\#kG$ for special data of finite Cartan type.

Theorem 3.5. *Let $\mathcal{D}$ be a special datum of finite Cartan type. Then*

$$U(\mathcal{D}, f) = R(\mathcal{D})\#kG/(x_\alpha^{N_\alpha} + u_\alpha(f), [x_i, x_j]_c + f(z_{ij})(1 - g_i g_j) | \alpha \in \Phi^+, i \not\sim j)$$

for $f \in \mathrm{Vect}_G(L(\mathcal{D}), k)$ are the liftings of $\mathcal{B}(V)\#kG = U(\mathcal{D}, \varepsilon)$. Moreover, all these liftings are monoidally Morita-Takeuchi equivalent.

Proof. Clearly, $U(\mathcal{D}, f)$ is a lifting of $\mathcal{B}(V)\#kG$ for the root vector parameters $\left\{ \mu_\alpha = f(x_\alpha^{N_\alpha} \middle| \alpha \in \Phi^+ \right\}$ and the linking parameters $\{\, \lambda_{ij} = f([x_i, x_j]_c) |\, i \not\sim j\}$. By [AS] all liftings of $\mathcal{B}(V)\#kG$ are of that form. To proof the last assertion let in 2.3 $H = kG$, $K = K\mathcal{D}$, $f \in \mathrm{Alg}_G(K, k)$. Then the ideal $I = (x_\alpha^{n_\alpha}, [x_i, x_j]_c | \alpha \in \Phi^+, i \not\sim j)$ and $J = \Theta(f)(I) = (x_\alpha^{N_\alpha} + u_\alpha(f), [x_i, x_j]_c + f(z_{ij})(1 - g_i g_j) | \alpha \in \Phi^+, i \not\sim j)$ of $K\#kG$ are conjugate. By 2.3 the quotient Hopf algebras $U(\mathcal{D}, \varepsilon)$ and $U(\mathcal{D}, f)$ of $R(\mathcal{D})\#kG$ are monoidally Morita-Takeuchi equivalent. The additional condition $(R\#kG)/(\mathrm{res}^{-1}(f) * (I\#kG)) \neq 0$ is verified in ([Ma2], Appendix). □

4. Cocycle deformations and cohomology

In this section we describe liftings of special crossed modules V over finite abelian groups in terms of cocycle deformations of $B(V)\#kG$, and determine the infinitesimal part of the deformations by means of Hochschild cohomology.

4.1. **Cocycle deformations.** A normalized 2-cocycle $\sigma \colon A \otimes A \to k$ on a Hopf algebra A is a convolution invertible linear map such that

$$(\varepsilon \otimes \sigma) * \sigma(1 \otimes m) = (\sigma \otimes \varepsilon) * \sigma(m \otimes 1)$$

and $\sigma(\iota \otimes 1) = \varepsilon = \sigma(1 \otimes \iota)$. The deformed multiplication

$$m_\sigma = \sigma * m * \sigma^{-1} \colon A \otimes A \to A$$

and antipode

$$s_\sigma = \sigma * s * \sigma^{-1} \colon A \to A$$

on A, together with the original unit, counit and comultiplication define a new Hopf algebra structure on H which we denote by A_σ. If A is $\mathbf{N}$-graded then $\sigma = \sum_{i=0}^{\infty} \sigma_i$, where $\sigma_j \colon A \otimes A \to k$ is the uniquely determined component of

degree $-j$ and $\sigma_0 = \varepsilon$. This corresponds to a convolution invertible normalized 2-cocycle

$$\sigma(t) = \sum_{i=0}^{\infty} \sigma_i t^i \colon A \otimes A \to k[[t]].$$

The convolution inverse $\sigma^{-1}(t) = \sum_{i=0}^{\infty} \eta_i t^i \colon A \otimes A \to k[[t]]$ is determined by $\sigma(t) * \sigma^{-1}(t) = \varepsilon = \sigma^{-1}(t) * \sigma(t)$, that is by

$$\sum_{i+j=l} \sigma_i * \eta_j = \delta_0^l = \sum_{i+j=l} \eta_i * \sigma_j.$$

The cocycle condition $(\varepsilon \otimes \sigma(t)) * \sigma(t)(1 \otimes m) = (\sigma(t) \otimes \varepsilon) * \sigma(t)(m \otimes 1)$ implies that

$$\sum_{i+j=l} (\varepsilon \otimes \sigma_i) * \sigma_j(1 \otimes m) = \sum_{i+j=l} (\sigma_i \otimes \varepsilon) * \sigma_j(m \otimes 1)$$

for all $l \geq 0$. In particular, if s is the least positive integer for which $\sigma_s \neq 0$ then $\eta_s = -\sigma_s$ and

$$\varepsilon \otimes \sigma_s + \sigma_s(1 \otimes m) = \sigma_s \otimes \varepsilon + \sigma_s(m \otimes 1)$$

so that $\sigma_s \colon A \otimes A \to k$ is a Hochschild 2-cocycle. The infinitesimal part mod (t^{s+1}) of $\sigma(t)$ and of $m_{\sigma(t)}$ are

$$\varepsilon \otimes \varepsilon + \sigma_s t^s \colon A \otimes A \to k[t]/(t^{s+1})$$

and

$$m_{\sigma(t)} = m + (\sigma_s * m - m * \sigma_s)t^s \colon A \otimes A \to A[t]/(t^{s+1}),$$

respectively, where

$$\phi = \sigma_s * m - m * \sigma_s \colon A \otimes A \to A$$

is a normalized Hochschild 2-cocycle.

Dually, a normalized 2-cocycle $\sigma \colon k \to A \otimes A$ is a convolution invertible linear map such that

$$(\iota \otimes \sigma) * (1 \otimes \Delta)\sigma = (\sigma \otimes \iota) * (\Delta \otimes 1)\sigma$$

and

$$(\varepsilon \otimes 1)\sigma = \iota = (1 \otimes \varepsilon)\sigma.$$

Then $H^\sigma = (H, m, \iota, \Delta^\sigma, \varepsilon)$ with the deformed comultiplication

$$\Delta^\sigma = \sigma * \Delta * \sigma^{-1} \colon A \to A \otimes A$$

is again a Hopf algebra. If A is $(-\mathbf{N})$-graded then $\sigma = \sum_{i=0}^{\infty} \sigma_i$, where $\sigma_j \colon A \otimes A \to k$ is the uniquely determined component of degree $-j$, corresponds to an invertible normalized 2-cocycle $\sigma(t) = \sum_{i=0}^{\infty} \sigma_i t^i \colon k \to A \otimes A[[t]]$.

Theorem 4.1. [Sch, Corollary 5.9] *If two Hopf algebras A and A' are cocycle deformations of each other, then they are monoidally Morita-Takeuchi equivalent. The converse is true if A and A' are finite dimensional.*

Suppose now that V is a crossed kG-module of special finite Cartan type, $\mathcal{A}(V)$ the free braided algebra and $\mathcal{A}(V)\#kG$ its bosonization. If I is the ideal of $\mathcal{A}(V)$ generated by the subset

$$S = \left\{ ad^{1-a_{ij}} x_i(x_j) \middle| \, i \sim j \right\} \cup \left\{ x_\alpha^{N_\alpha} \middle| \, \alpha \in \Phi^+ \right\} \cup \left\{ [x_i, x_j]_c \middle| \, i \not\sim j \right\}$$

then $\mathcal{A}(V)/I = \mathcal{B}(V)$ is the Nichols algebra. The subalgebra K of $\mathcal{A}(V)$ generated by S is a Hopf subalgebra [AS], [CP, Proposition 9.2.1]. Then $K\#kG$ is the Hopf subalgebra of $\mathcal{A}(V)\#kG$ generated by S and G.

Lemma 4.2. *The injective group homomorphism*

$$\phi\colon \operatorname{Alg}_G(K, k) \to \operatorname{Alg}(K\#kG, k)$$

given by $\phi(f)(x\#g) = f(x)$ *has image*

$$\widetilde{\operatorname{Alg}(K\#kG, k)} = \left\{ f \in \operatorname{Alg}(K\#kG, k) | \, f_{|kG} = \varepsilon \right\}$$

and

$$\operatorname{adj}\colon \operatorname{Alg}_G(K, k) \to \operatorname{Aut}(K\#kG)$$

has its image in the subgroup

$$\widetilde{Aut(K\#kG)} = \left\{ f \in \operatorname{Aut}(K\#kG) | \, f_{|kG} = \varepsilon \right\}.$$

Moreover, if V *is of special finite Cartan type then* $f(ad^{1-a_{ij}} x_i(x_j)) = 0$ *for* $i \sim j$ *and for every* $f \in \operatorname{Alg}_G(K, k)$.

Proof. If $f \in \operatorname{Alg}(K\#kG, k)$ then $f(ad^{1-a_{ij}} x_i(x_j)) = f(g \cdot ad^{1-a_{ij}} x_i(x_j) g^{-1}) = \chi_i(g)^{1-a_{ij}} \chi_j(g)$, $f(x_\alpha^N) = f(g x_\alpha^{N_\alpha} g^{-1}) = \chi_\alpha^{N_\alpha}(g) f(x_\alpha^N)$ and $f([x_i, x_j]_c) = f(g[x_i, x_j]_c g^{-1}) = \chi_i(g)\chi_j(g) f([x_i, x_j]_c)$, so that $f(g \cdot ad^{1-a_{ij}} x_i(x_j)) = 0$ if $\chi_i^{1-a_{ij}} \chi_j \neq 0$, $f(x_\alpha^{N_\alpha}) = 0$ if $\chi_\alpha^{N_\alpha} \neq \varepsilon$ and $f([x_i, x_j]_c) = 0$ if $\chi_i \chi_j \neq \varepsilon$. □

The theorem above can now be applied to the situation in Section 3 to show that all 'liftings' of a crossed kG-module of special finite Cartan type are cocycle deformations of each other. The special case of quantum linear spaces has been studied by Masuoka [Ma], and that of a crossed kG-module corresponding to a finite number of copies of type A_n by Didt [Di].

Theorem 4.3. *Let* G *be a finite abelian group,* V *a crossed* kG*-module of special finite Cartan type,* $\mathcal{B}(V)$ *its Nichols algebra with bosonization* $A = \mathcal{B}(V)\#kG$*. Then:*

(1) *All liftings of* A *are monoidally Morita-Takeuchi equivalent, i.e: their comodule categories are monoidally equivalent, or equivalently,*
(2) *all liftings of* A *are cocycle deformations of each other.*

Proof. The main theorem 3.5 at the end the last section says that $\mathcal{B}(V)\#kG \cong U(\mathcal{D}, \varepsilon) \cong R(\mathcal{D})\#kG/(I)$ for a Hopf ideal I in the Hopf subalgebra $K(\mathcal{D})\#kG$ of $R(\mathcal{D})\#kG$, that its liftings are of the form $U(\mathcal{D}, f) \cong R(\mathcal{D})\#kG/(I_f)$ for a conjugate Hopf ideal I_f, where $f \in \operatorname{Alg}_G(K(\mathcal{D}, k), k) \cong \widetilde{\operatorname{Alg}}(K(\mathcal{D})\#kG, k)$, and

that they are all Morita-Takeuchi equivalent. Thus, Schauenburg's result applies, so that all these liftings are cocycle deformations of each other. □

Corollary 4.4. *Let H be a finite dimensional pointed Hopf algebra with abelian group of points $G(H) = G$ and assume that the order of G has no prime divisors < 11. Then:*

- *H and $\operatorname{gr}_c(H)$ are Morita Takeuchi equivalent, or equivalently,*
- *H is a cocycle deformation of $\operatorname{gr}_c(H)$.*

Proof. Under the present assumptions the Classification Theorem [AS] asserts that $\operatorname{gr}_c(H) \cong B(V)\#kG$ for a crossed kG-module V of special finite Cartan type, and hence the previous theorem applies. □

In the case at hand $A = B(V)\#kG$ and the condition that $\operatorname{gr}_c A^\sigma \cong A$ implies that the cocycle $\sigma\colon A \otimes A \to k$ is G-invariant, since $m_\sigma(x \otimes g) = m(x \otimes g)$ and $m_\sigma(x \otimes g) = m(x \otimes g)$ for all $g \in G$, so that

$$\sigma(x \otimes g) = \varepsilon(x) = \sigma(g \otimes x)$$

for all $g \in G$. The cocycle conditions then imply that

$$\sigma(x \otimes yg) = \sigma(x \otimes y)\ ,\ \sigma(xg \otimes y) = \sigma(x \otimes gy)\ ,\ \sigma(gx \otimes y) = \sigma(x \otimes y),$$

which means that σ factors through $A \otimes_{kG} A$ and also that the cocycle really comes from a convolution invertible G-invariant 2-cocycle

$$\nu\colon B(V) \otimes B(V) \to k.$$

In fact, the restriction of a G-invariant 2-cocycle $\sigma\colon A \otimes A \to k$ restricts to a G invariant 2-cocycle on $B(V) \otimes B(V)$ and the map

$$\Psi\colon Z^2_G(B(V), k) \to Z^2_G(A, k),$$

defined by $\Psi(\nu)(x\#g \otimes x'\#g') = \nu(x \otimes g(x'))$, is inverse to the restriction map. This map is of degree zero and therefore also defines a bijection between the associated sets of formal cocycles

$$\Psi\colon Z^2_G(B(V), k[[t]]) \to Z^2_G(A, k[[t]]),$$

and the infinitesimal parts, which are Hochschild cocycles.

4.2. **Exponential map.** It is in general very hard to give explicit examples of multiplicative cocycles. One somewhat accessible family consists of bicharacters. Below we give another idea which can sometimes be used.

Note that if $B = \oplus_{n=0}^\infty B_n$ is a graded bialgebra, and $f\colon B \to k$ is a linear map such that $f|_{B_0} = 0$, then

$$e^f = \sum_{i=0}^\infty \frac{f^{*i}}{i!}\colon B \to k$$

is a well defined convolution invertible map with convolution inverse e^{-f}. When $f\colon B \otimes B \to k$ is a Hochschild cocycle such that $f|_{B \otimes B_0 + B_0 \otimes B} = 0$, then 'often'

$e^f : B\otimes B \to k$ will be a multiplicative cocycle. For instance this happens whenever $f(1\otimes m)$ and $f(m\otimes 1)$ commute (with respect to the convolution product) with $\varepsilon\otimes f$ and $f\otimes\varepsilon$, respectively. Also note that if $f * f = 0$, then $e^f = \varepsilon + f$.

From now on assume B is obtained as a bosonization of a quantum linear space. More precisely $B = \left\langle G, x_1, \dots, x_\theta \,|\, gx_i = \chi_i(g)x_i g, x_i x_j = \chi_j(g_i)x_j x_i, x_i^{N_i} = 0\right\rangle$. Here $\chi_1, \dots, \chi_\theta \in \widehat{G}$, $g_1, \dots, g_\theta \in \Gamma$ are such that $\chi_i(g_j)\chi_j(g_i) = 1$ for $i \neq j$. Number N_i is the order of $\chi_i(g_i)$. We abbreviate $q_{i,j} = \chi_i(g_j)$. Then $\zeta_i : B\otimes B \to k$, given by

$$\zeta_i(xg, yh) = \begin{cases} \chi_i^{b_i}(g), & \text{if } x = x_i^{a_i}, y = x_i^{b_i}, a_i + b_i = N_i \\ 0, & \text{otherwise} \end{cases}$$

for $x = x_1^{a_1}\dots x_\theta^{a_\theta}$ and $y = x_1^{b_1}\dots x_\theta^{b_\theta}$ (see Corollary 4.8) is a Hochschild cocycle. Moreover, each of the sets

$$A_l = \{\,(\varepsilon\otimes\zeta_i), \zeta_i(1\otimes m) |\, 1 \le i \le \theta\}$$

and

$$A_r = \{\,(\zeta_i\otimes\varepsilon), \zeta_i(m\otimes 1) |\, 1 \le i \le \theta\}$$

is a commutative set (for the convolution product). We sketch the proof for A_l (the proof for A_r is symmetric). Maps $\zeta_i(1\otimes m)$ and $\zeta_j(1\otimes m)$ commute since ζ_i and ζ_j do. Same goes for $\varepsilon\otimes\zeta_i$ and $\varepsilon\otimes\zeta_j$. Hence it is sufficient to prove that for all i, j we have

$$(\varepsilon\otimes\zeta_i) * (\zeta_j(1\otimes m)) = (\zeta_j(1\otimes m)) * (\varepsilon\otimes\zeta_i).$$

If $i \neq j$, this is immediate. For $i = j$ note that both left and right hand side can be nonzero only at PBW elements of the form $x_i^r f\otimes x_i^s g\otimes x_i^p h \in B\otimes B\otimes B$, with $r + s + p = 2N_i$. Without loss of generality assume that $f = g = h = 1$. In this case the left hand side evaluates to

$$\sum_{u+v=N_i} \binom{s}{u}_{q_{ii}} \binom{p}{v}_{q_{ii}} q_{ii}^{u(p-v)} = 1$$

and the right hand side is

$$\sum_{u+v=N_i-r} \binom{s}{u}_{q_{ii}} \binom{p}{v}_{q_{ii}} q_{ii}^{u(p-v)} = 1.$$

Thus if f is any map in the linear span $Span_k\{\zeta_i\}$, then e^f is a multiplicative cocycle.

This idea is illustrated in some of the examples given in Section 5.3.

4.3. **The standard cosimplicial algebra complex and cohomology of braided Hopf algebras.** The 'multiplicative' cocycles above and the 'additive' Hochschild cocycles can in principle be computed from the normalized standard cosimplicial complex associated with the standard comonad $A \otimes -$ on the category of A-bimodules. The relevant part of that complex with coefficients in the A-bimodule M is

$$\mathrm{Hom}(k, M) \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \mathrm{Hom}(A, M) \overset{\partial_0}{\underset{\partial_1,\ \partial_2}{\rightrightarrows}} \mathrm{Hom}(A^2, M) \overset{\partial_0}{\underset{\partial_1,\ \partial_2,\ \partial_3}{\rightrightarrows}} \mathrm{Hom}(A^3, M)$$

with coface maps $\partial_i \colon \mathrm{Hom}(A^n, M) \to \mathrm{Hom}(A^{n+1}, M)$ given by

$$\partial_i(f) = \begin{cases} \mu_l(1 \otimes f) & \text{, if i = 0} \\ f(1^{i-1} \otimes m \otimes 1^{n-i-1}) & \text{, if } 1 \leq \text{i} \leq \text{n} - 1 \\ (f \otimes 1)\mu_r & \text{, if i = n} \end{cases}$$

and codegeneracy maps $s_i \colon \mathrm{Hom}(A^{n+1}, M) \to \mathrm{Hom}(A^n, M)$, $s_i f = f(1^i \otimes \iota \otimes 1^{n-i})$, where $\iota \colon k \to A$ is the unit. Hochschild (or the 'additive') cohomology $H^*(R, M)$ is the cohomology of the associated cochain complex with the alternating sum differentials $\partial = \sum_{i=0}^{n} (-1)^i \partial_i \colon \mathrm{Hom}(A^n, M) \to \mathrm{Hom}(A^{n+1}, M)$, so that

$$\begin{aligned} \partial f(a_1 \otimes \ldots \otimes a_{n+1}) &= a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \\ &+ \sum_{i+1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1}. \end{aligned}$$

If $M = k$, the trivial A-bimodule, then the cosimplicial complex is a cosimplicial algebra under convolution. Apply the group of units functor to this cosimplicial algebra to get a generally non-abelian cosimplicial group. Then

$$Z^1(A, k) = \{ f \in \mathrm{Hom}(A, k) |\, \partial_2(f) * \partial_0(f) = \partial_1(f) \} = \mathrm{Alg}(A, k)$$

is the group of 'multiplicative' 1-cocycles, while

$$\begin{aligned} Z^2(A, k) &= \{ f \in \mathrm{Hom}(A \otimes A, k) |\, \partial_3(f) * \partial_1(f) = \partial_0(f) * \partial_2(f) \} \\ &= \left\{ f \in \mathrm{Hom}(A^2, k) \middle| \, f(x_1, y_1) f(x_2 y_2, z) = f(y_1, z_1) f(x, y_2 z_2) \right\} \end{aligned}$$

is the set of 'multiplicative' 2-cocycles. In case A is cocommutative, the cosimplicial group is abelian and from the associated cochain complex with the alternating convolution product differentials one gets Sweedler cohomology.

This theory also works for a braided algebra in the category of crossed H-modules when the tensor products are taken in the braided sense.

Proposition 4.5. *If A and A' are finite dimensional (braided) algebras then*

$$H^*(A, M) \otimes H^*(A', M') \cong H^*(A \otimes A', M \otimes M')$$

Proof. The Bar-Resolution $B(A, M)$ of the A-bimodule M, with differential

$$d\colon B_{n+1}(A, M) = A \otimes A^n \otimes M \to A \otimes A^{n-1} \otimes M = B_n(A, M)$$

given by

$$d(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes m) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n \otimes m,$$

is a k-split relatively free resolution of M. The Hochschild cohomology of A with coefficients in M is defined by $H^*(A, M) = H^*({}_A \operatorname{Hom}_A(B(A, A), M))$. Both, $B(A, M) \otimes B(A', M')$ and $B(A \otimes A', M \otimes M')$ are k-split relatively free resolutions of the $A \otimes A'$-bimodule $M \otimes M'$. By the comparison theorem the two chain complexes are chain equivalent. Such a chain equivalence is given by the (braided) version of the Alexander-Whitney map. By the (braided) version of the Kunneth Theorem there is a natural map

$$H^*B(A, M) \otimes H^*B(A', M') \to H^*(B(A, M) \otimes B(A', M')) \cong H^*B(A \otimes A', M \otimes M')$$

which is an isomorphism when either A or A' is finite dimensional. □

This result can be applied to Nichols algebras of certain finite dimensional Yetter-Drinfel'd modules over abelian groups.

Theorem 4.6. *If $V = \oplus_{J \in \mathcal{X}} V_J$ is the crossed kG-module of a special datum $\mathcal{D}$ of finite Cartan type, where $\mathcal{X}$ is the set of connected components of the Dynkin diagram, then*

$$B(V) \cong \otimes_{J \in \mathcal{X}} B(V_J)$$

as a braided Hopf algebra and

$$H^*(B(V), k) \cong \otimes_{J \in \mathcal{X}} H^*(B(V_J), k)$$

as a graded vector space.

□

Corollary 4.7. *If $V = \oplus_{i=1}^{t} k x_i$ is a quantum linear space over an abelian group G then $B(V) \cong B_1 \otimes B_2 \otimes \ldots \otimes B_t$, where $B_i = B(kx_i) \cong k[x_i]/(x_i^{n_i})$. Moreover,*

$$H^*(B(V)) \cong H^*(B_1) \otimes H^*(B_2) \otimes \ldots \otimes H^*(B_t)$$

with $H^j(B_i, B_i) \cong k[x_i]/(x_i^{n_i - 1})$ and $H^j(B_i, k) \cong k$.

□

Remark 4.8. *Note that if $A = B(kx) = k[x]/(x^n)$, then*

$$H^0(A, k) = k \text{ and } H^1(A, k) = \operatorname{Vect}(A^+/(A^+)^2, k).$$

A normalized 2-cocycle $f\colon A^+ \otimes A^+ \to k$ is a linear map satisfying $f(x^i \otimes x^j) = f(x^k \otimes x^l)$ whenever $i + j = k + l$, so that $Z^2(A, k) = \oplus k f_l$, where $f_l(x^i \otimes x^j) = 1$ if $i + j = l$ and $f_l(x^i \otimes x^j) = 0$ otherwise. If $f(x^i \otimes x^j) = 0$ for $i + j = n$ then $f = \delta g$,

where $g(x^{i+j}) = f(x^i \otimes x^j)$, so that $H^2(A,k) = Z^2(A,k)/B^2(A,k)$ is represented by f_n.

4.4. **The equivariant cohomology.** The G-invariant Hochschild cocycles are described via the cosimplicial complex of G-invariant elements in the standard complex. The commutative 'pushout-pullback' square of (braided) Hopf algebras in Section 2

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & R \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ k & \xrightarrow{\iota} & B \end{array}$$

induces a square of cosimplicial algebras

$$\begin{array}{ccccccc}
\mathrm{Hom}_G(k,k) & \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} & \mathrm{Hom}_G(B,k) & \substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}} & \mathrm{Hom}_G(B^2,k) & \substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}\\ \xrightarrow{\partial_3}} & \mathrm{Hom}_G(B^3,M) \\
\| & & \downarrow \pi^* & & \downarrow (\pi^2)^* & & \downarrow (\pi^3)^* \\
\mathrm{Hom}_G(k,k) & \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} & \mathrm{Hom}_G(R,k) & \substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}} & \mathrm{Hom}_G(R^2,k) & \substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}\\ \xrightarrow{\partial_3}} & \mathrm{Hom}_G(R^3,M) \\
\| & & \downarrow \kappa^* & & \downarrow (\kappa^2)^* & & \downarrow (\kappa^3)^* \\
\mathrm{Hom}_G(k,k) & \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} & \mathrm{Hom}_G(K,k) & \substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}} & \mathrm{Hom}_G(K^2,k) & \substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}\\ \xrightarrow{\partial_3}} & \mathrm{Hom}_G(A^3,M)
\end{array}$$

where the trivial part has been omitted. Here is a equivariant analog of the 5-term sequence, which allows a direct calculation of the infinitesimal deformation cocycle associated with the equivariant algebra map $f \in \mathrm{Alg}_G(K,k)$.

Theorem 4.9. *There is an exact sequence*

$$0 \to H^1_G(B,k) \xrightarrow{\pi^*} H^1_G(R,k) \xrightarrow{\kappa^*} H^1_G(K,k) \xrightarrow{\delta} H^2_G(B,k) \xrightarrow{(\pi\otimes\pi)^*} H^2_G(R,k)$$

Proof. To construct $\partial : H^1_G(K,k) \to H^2_G(B,k)$ observe first that

$$\mathrm{Der}_G(K,k) = H^1_G(K,k) = Z^1_G(K,k) = \{f \in \mathrm{Hom}_G(K,k) | \partial^1 f = \partial^0 f + \partial^2 f\}.$$

Choose a K-bimodule retraction $u : R \to K$ for $\kappa : K \to R$ so that $u\kappa = 1_K$ and $\varepsilon_K u = \varepsilon_R$. Then $(\kappa \otimes \kappa)^* \partial^i u^* = \partial^i \kappa^* u^* = \partial^i$ for $i = 0, 1, 2$. It also follows that

$$\partial u^* f(K^+ R \otimes R + R \otimes RK^+) = 0$$

for any $f \in Z^1_G(K,k)$, since u is a K-bimodule map, so that

$$\begin{aligned} \partial u^* f(xr \otimes r') &= \varepsilon(xr) u^* f(r') - u^* f(xrr') + u^* f(xr)\varepsilon(r') \\ &= -f(xu(rr')) + f(xu(r))\varepsilon(r') \\ &= -f(x)\varepsilon u(rr') + f(x)\varepsilon u(r)\varepsilon(r') = 0 \end{aligned}$$

and similarly $\partial u^* f(r\otimes r'x) = 0$. This means that the 2-cocycle $\partial u^* f : R\otimes R \to k$ factors uniquely through $\pi\otimes\pi : R\otimes R \to B\otimes B$, i.e: there exists a unique 2-cocycle $\bar{f} : B\otimes B \to k$ such that $(\pi\otimes\pi)^*\bar{f} = \partial u^* f$. So define

$$\delta : H^1_G(K, k) \to H^2_G(B, k)$$

by $\delta(f) = [\bar{f}]$, the cohomology class of $\bar{f}$.

Exactness at $H^1_G(B, k)$: It is clear that $\pi^* : H^1_G(B, k) \to H^1_G(R, k)$ is injective, since π is surjective, and that $\kappa^*\pi^* = (\pi\kappa)^* = (\iota\varepsilon)^* = \varepsilon^*\iota^*$ is the trivial map, since $f(1) = 0$ for $f \in Z^1_G(B, k)$.

Exactness at $H^1_G(R, k)$: Suppose that $f \in Z^1_G(R, k)$ and $\kappa^*(f) = 0$. Then $f(RK^+R) = 0$, since $f(rxr') = \varepsilon(r)f(xr') + f(r)\varepsilon(xr') = \varepsilon(r)\varepsilon(x)f(r') + \varepsilon(r)f(x)\varepsilon(r') + f(r)\varepsilon(xr') = 0$ for $x \in K^+$. Hence, there is a unique $f' \in \mathrm{Hom}_G(B, k)$ such that $\pi^*(f') = f$. Moreover, $0 = \partial f = \partial\pi^* f' = (\pi^*\otimes\pi^*)\partial f'$, so that $\partial f' = 0$, since $(\pi\otimes\pi)^*$ is injective.

Exactness at $H^1(K, k)$: First show that $\delta\kappa^* = 0$. If $f \in Z^1_G(R, k)$, then $f(1) = 0$ and $\delta\kappa^* f = [\tilde{f}] \in H^2_G(B.k)$ with $\tilde{f} \in Z^2_G(B.k)$, and $(\pi\otimes\pi)^*\tilde{f} = \partial u^*\kappa^* f \in B^2_G(B, k)$ Moreover, $0 = \partial f(\kappa(x)\otimes r) = -f(\kappa(x)r) + f\kappa(x)\varepsilon(r)$, so that $(f - u^*\kappa^* f)(\kappa(x)r) = f(\kappa(x)r - f(\kappa(x)\kappa u(r)) = f\kappa(x)\varepsilon(r) - f\kappa(x)\varepsilon\kappa u(r) = 0$. Thus $f - u^*\kappa^* f$ factors uniquely through $\pi : R \to B$, i.e: $\pi^* f' = u^*\kappa^* f - f$ for a unique $f' \in \mathrm{Hom}_G(B, k)$. But then $(\pi\otimes\pi)^*\partial f' = \partial\pi^* f' = \partial(u^*\kappa^* f - f) - \partial u^*\kappa^* f = (\pi\otimes\pi)^*\tilde{f}$, and therefore $\tilde{f} = \partial f' \in B^2_G(B, k)$, since $(\pi\otimes\pi)^*$ is injective.

Now, if $\delta(f) = [\bar{f}] = 0$ for a given $f \in Z^1_G(K, k)$, then $\bar{f} = \partial f'$ for some $f' \in \mathrm{Hom}_G(B^+, k)$, and $\partial\pi^* f' = (\pi\otimes\pi)^*\partial f' = (\pi\otimes\pi)^*\bar{f} = \partial u^* f$, so that $u^* f - \pi^* f' \in Z^1_G(R, k)$. Then $\kappa^*(u^* f - \pi^* f') = f - (\pi\kappa)^* f' = f \in Z^1_G(K, k)$.

Exactness at $H^2_G(B, k)$: Finally, if $(\pi\otimes\pi)^*[f] = 0$ for a given $[f] \in H^2_G(B, k)$ then $(\pi\otimes\pi)^* f = \partial f'$ for some $f' \in \mathrm{Hom}_G(R^+, k)$. Moreover, $\partial f'(\kappa(x)\otimes r) = \varepsilon\kappa(x)f'(r) - f'(\kappa(x)r) + f'\kappa(x)\varepsilon(r) = (\pi\otimes\pi)^* f(\kappa(x)\otimes r) = f(\pi\kappa(x)\otimes\pi(r)) = f(\iota_B\varepsilon_K(x)\otimes\pi(r))$. Thus, if $x \in K^+$ then $f'(\kappa(x)r) = f'\kappa(x)\varepsilon(r)$ and $(f' - u^*\kappa^* f')(\kappa(x)r) = f'(\kappa(x)r) - f'(\kappa(x)\kappa u(r)) = f'\kappa(x)\varepsilon(r) - f'\kappa(x)\varepsilon\kappa u(r) = 0$, so that K^+R is in the kernel of $(f' - u^*\kappa^* f')$. It follows that there is a unique $f'' : B \to k$ such that $\pi^* f'' = f' - u^*\kappa^* f'$. Then $(\pi\otimes\pi)^*(f - \partial f'') = (\pi\otimes\pi)^* f - \partial\pi^* f'' = \partial u^*\kappa^* f'$, which means that $[f] = [f - \partial f''] = \delta(\kappa^* f')$. $\square$

The connecting map $\delta : H^1_G(K, k) \to H^2_G(B, k)$ can be used to describe the infinitesimal part of the 'multiplicative' cocycles $\sigma : B\otimes B \to k$ in terms of the algebra map $f \in \mathrm{Alg}_G(K, k)$ taking into account that $L(\mathcal{D}) \cong K^+/(K^+)^2$ and $H^1_G(K, k) \cong \mathrm{Hom}_G(K^+/(K^+)^2, k)$.

If $A = B\#kG$ then the collection of isomorphisms

$$\Psi_n : \mathrm{Hom}_G(B^n, k) \to {}_G\,\mathrm{Hom}_G(A^n, k),$$

given by $\Psi_n(f)(b_1 g 1, b_2 g_2, \ldots, b_n g_n) = f(b_1, g_1(b_2), \ldots, g_1 g_2 \ldots g_{n-1}(b_n)$ and $\Psi^{-1} f'(b_1, b_2, \ldots, b_n) = f'(b_1 1, b_2 1, \ldots, b_n 1)$, defines an isomorphism of complexes,

which induces an isomorphism in cohomology

$$\Psi^* : H^*_G(B,k) \to {}_G H^*_G(B\#kG,k).$$

The image of the composite

$$\Psi^2\delta : H^1_G(K,k) \to H^2_G(B,k) \to {}_g H^2_G(B\#kG,k)$$

consists of the infinitesimal parts of of the 'multiplicative' cocycles. If $\zeta_F = \Psi^2\delta(f)$ then $(\zeta_f * m - m * \zeta_f) \in H^2(A,A)$ and $m + (\zeta_f * m - m * \zeta_f) : A \otimes A \to A$ is the infinitesimal part of the cocycle deformation associated with $f \in \mathrm{Alg}_G(K,k)$.

5. Liftings and deformations

The formal cocycle deformations of the previous section are in particular formal deformations in the sense of [GS, DCY]. The subject of this section is the relation between formal deformations, liftings and Hochschild cohomology of (braided) Hopf algebras.

5.1. **Deformations of (graded) bialgebras.** The formal deformation of a (graded) bialgebra $(A, m, \Delta, \iota, \varepsilon)$ is a bialgebra structure $(A[[t]], m(t), \Delta(t), \iota, \varepsilon)$ on the free $k[[t]]$-module $A[[t]] = A \otimes k[[t]]$, such that $m(0) = m$ and $\Delta(0) = \Delta$. Here $m(t) = \sum i \geq 0 \mu_i t^i$ and $\Delta(t) = \sum_{i \geq 0} \delta_i t^i$ are determined by sequences of linear maps $\mu_i \colon A \otimes A \to A$ and $\delta_i \colon A \to A \otimes A$. An l-deformation of A is a bialgebra structure on the free $k[[t]]/(t^{l+1})$-module $A_l = A[[t]]/(t^{l+1})$. The associativity, coassociativity and compatibility conditions are

(1) Associativity: $\sum_{r+s=i} \mu_r(\mu_s \otimes 1) = \sum_{r+s=i} \mu_r(1 \otimes \mu_s)$,
(2) Coassociativity: $\sum_{r+s=i}(\delta_r \otimes 1)\delta_s = \sum_{r+s=i}(1 \otimes \delta_r)\delta_s$,
(3) Compatibility: $\sum_{r+s=i} \delta_r \mu_s = \sum_{r+s+u+v=i}(\mu_r \otimes \mu_s)\tau_{23}(\delta_u \otimes \delta_v)$.

In particular for infinitesimal deformations, the case $l = 1$, these are 2-cocycle conditions in the bialgebra cohomology.

An isomorphism of l-deformations is an isomorphism of $k[[t]]/(t^{l+1})$-bialgebras

$$f \colon (A_l, m_l, \Delta_l) \to (A_l, m'_l, \Delta'_l)$$

such that $\iota^*(f) = id_A$. Such an isomorphism is of the form $f = \sum_{i \geq 0} f_i T^i$ for a sequence of maps $f_i \colon H \to H$ satisfying the conditions

(4) $\sum_{r+s=i} f_r \mu_s = \sum_{t+u+v=i} \mu'_t(f_u \otimes f_v)$
(5) $\sum_{r+s=i} \delta'_r f_s = \sum_{u+v+t=i}(f_u \otimes f_v)\delta_t$

required by the fact that f is a $k[[t]]/(t^{l+1})$-bialgebra map. The set of isomorphism classes of l-deformations of A will be denoted by $\mathrm{Def}_l(A)$. The projection $k[[t]]/(t^{l+1}) \to k[[t]]/(t^l$ induces a restriction map $\mathrm{res}_l \colon \mathrm{Def}_{l+1}(A) \to \mathrm{Def}_l(A)$ and

$$\mathrm{Def}(A) = \lim_{\leftarrow} \mathrm{Def}_l(A)$$

is the set of isomorphism classes of formal deformations (∞-deformations) of A.

Theorem 5.1. [Gr2] *The restriction map* $\mathrm{res}_l\colon \mathrm{Def}_{l+1}(A) \to \mathrm{Def}_l(A)$ *fits into an exact sequence of pointed sets*

$$H^2(A,A) \longrightarrow \mathrm{Def}_{l+1}(A) \xrightarrow{\mathrm{res}_l} \mathrm{Def}_l(A) \xrightarrow{\mathrm{obs}_l} H^3(A,A)$$

for $l \geq 0$*. In particular:*

- $H^2(A,A) \cong \mathrm{Def}_1(A)$ *is an abelian group,*
- *Every formal deformation of* A *is trivial if and only if* $H^2(A,A) = 0$*,*
- *If* $H^3(A,A) = 0$ *then every infinitesimal deformation can be extended to a formal deformation.*

Proof. (Sketch) If two $(l+1)$-deformations restrict to the same l-deformation then they differ by a pair of compatible 2-cocycles $(\mu_{l+1} - \mu'_{l+1}, \delta_{l+1} - \delta'_{l+1})$. If (A_l, m_l, Δ_l) is an l-deformation then

$$\psi = (\sum_{i+j=l+1} \mu_i(\mu_j \otimes 1 - 1 \otimes \mu_j), \sum_{i+j=l+1} (\delta_i \otimes 1 - 1 \otimes \delta_i)\delta_j)$$

is a 3-cocycle, and $\mathrm{obs}_l(A_l, m_l, \Delta_l)$ is the cohomology class of ψ. Thus, if $\mathrm{obs}_l(A_l, m_l, \Delta_l) = 0$ then ψ is a 3-coboundary, that is $\psi = \partial(\mu_{l+1}, \delta_{l+1}) = (\mu_0(1 \otimes \mu_{l+1} - \mu_{l+1} \otimes 1) + \mu_{l+1}(1 \otimes \mu_0 - \mu_0 \otimes 1), (1 \otimes \delta_{l+1} - \delta_{l+1} \otimes 1)\delta_0 + (1 \otimes \delta_0 - \delta_0 \otimes 1)\delta_{l+1})$ for some $\mu_{l+1}\colon A \otimes A \otimes A \to A$ and $\delta_{l+1}\colon A \to A \otimes A \otimes A$, and $(A_l, m_l, \Delta_l) = \mathrm{res}_l(A_{l+1}, m_{l+1}, \Delta_{l+1})$. □

5.2. **Liftings of (graded) bialgebras.** If $A = \oplus_{n \geq 0} A_i$ is a graded bialgebra then it carries an ascending bialgebra filtration $A_c^{(j)} = \oplus_{i \leq j} A_i$ and a descending bialgebra filtration $A^r_{(j)} = \oplus_{i \geq j} A_i$. A lifting of a graded bialgebra A is a filtered Hopf algebra structure $K = (A, M, \Delta)$ on the vector space A such that $\mathrm{gr}_c K \cong A$. A co-lifting of A is a co-filtered bialgebra structure $G = (A, M, \Delta)$ such that $\mathrm{gr}_r(G) \cong A$. Let

$$\mathrm{Lift}(A) \quad \text{and} \quad \text{co-Lift}(A)$$

be the sets of equivalence classes of liftings and of co-liftings of A, respectively.

Theorem 5.2 (cf. [DCY])**.** *There are bijections* $\mathrm{Lift}(A) \cong \mathrm{Def}(A) \cong \text{co-Lift}(A)$.

Proof. We deal with the co-lifting part of the theorem. Let $G = (A, M, \Delta)$ be a co-lifting of the graded bialgebra H, so that $\mathrm{gr}_r G = H$. The multiplication and the comultiplication are maps of co-filtered vector spaces and they uniquely determine maps $\mu_r\colon A \otimes A \to A$ and $\delta_r : A \to A \otimes A$ of degree r for every $r \geq 0$, such that $M(a \otimes b) = \sum_{r \geq 0} \mu_r(a \otimes b)$ and $\Delta(c) = \sum_{r \geq 0} \delta_r(c)$. By associativity of M, coassociativity of Δ and compatibility of the two structure maps these linear maps satisfy exactly the conditions (1), (2) and (3) of the previous subsection. Now define a bialgebra $D(G) = (A[[t]], m_d, \Delta_d)$ over $k[[t]]$ by $m_d = \sum_{i \geq 0} \mu_i t^i$ and $\Delta_d = \sum_{i \geq 0} \delta_i t^i$. This gives a well-defined bijection

$$D\colon \text{co-Lift}(H) \to \mathrm{Def}(H)$$

since equivalent co-liftings are sent to isomorphic deformations. An isomorphism of co-liftings $f\colon G \to G'$ is a map of co-filtered bialgebras so that $f(a) = \sum_{r\geq 0} f_r(a)$ for uniquely determined linear maps $f_r\colon A \to A$ of degree r, which satisfy the conditions (4) and (5) of the previous subsection since f is a bialgebra map. The induced map $f_d\colon D(G) \to D(G')$, defined by $f_d = \sum_{i\geq 0} f_i T^i$, is an isomorphism of deformations. Similar arguments work for liftings [DCY], but now the linear maps μ_r, δ_r and f_r are of degree $-r$. □

Lemma 5.3 (cf. [GS], [MW]). *If $\zeta\colon A\otimes A \to k$ is a Hochschild cocycle of degree -n then the linear map $\mu = (\zeta\otimes m - m\otimes\zeta)\Delta_{A\otimes A}\colon A\otimes A\to A$ is of degree $-n$ and satisfies the cocycle condition*

$$m(\mu\otimes 1) + \mu(m\otimes 1) = m(1\otimes\mu) + \mu(1\otimes m),$$

so that $m_{\zeta,t} = m + \mu t^n\colon (A\otimes A)[t] \to A[t]$ is an infinitesimal deformation.

5.3. **Examples.** To illustrate the discussion above let us consider the case of one-dimensional crossed modules over a cyclic group.

1. Let $G = \langle g\rangle$ be a cyclic group of order np and let $V = kx$ be a 1-dimensional crossed G-module with action and coaction given by $gx = qx$ for a primitive n-th root of unity q and $\delta(x) = g\otimes x$. The braiding $c\colon V\otimes V \to V\otimes V$ is then determined by $c(x\otimes x) = qx\otimes x$. The braided Hopf algebra $\mathcal{A}(V)$ is the polynomial algebra $k[x]$ with comultiplication $\Delta(x^i) = \sum_{r+s=i}\binom{i}{r}_q x^r\otimes x^s$ in which x^n is primitive. The braided Hopf algebra $\mathcal{C}(V) = k\langle x\rangle$ is the divided power Hopf algebra with basis $\{x_i \,|\, i\geq 0\}$, comultiplication $\Delta(x_i) = \sum_{r+s=i} x_r\otimes x_s$ and multiplication $x_i x_j = \binom{i+j}{i}_q x_{i+j}$. The quantum symmetrizer $\mathcal{S} : \mathcal{A}(V) \to \mathcal{C}(V)$ is given by $\mathcal{S}(x^i) = \mathcal{S}(x)^i = i_q! x_i$. The Nichols algebra of V is $B(V) = \mathcal{A}(V)/(x^n) \cong \operatorname{im}\mathcal{S}$ and the Hopf algebra

$$\begin{aligned} A &= B(V)\# kG \\ &= \langle x, g | x^n = 0, g^{np} = 1, gx = qxg, \Delta(x) = x\otimes 1 + g\otimes x, \Delta(g) = g\otimes g\rangle \end{aligned}$$

is coradically graded.

The convolution invertible linear functional $\sigma\colon A\otimes A\to k$ defined by

$$\sigma(x^i g^u\otimes x^j g^v) = \begin{cases} 1, & \text{if } i+j=0; \\ 0, & \text{if } 0 < i+j < n; \\ aq^{ju}, & \text{if } i+j = n \end{cases}$$

is a cocycle of the form $\sigma = \varepsilon\otimes\varepsilon + \zeta$, where $\zeta(x^i g^u\otimes x^j g^v) = aq^{ju}\delta_n^{i+j}$ is a functional of degree $-n$ and $\zeta^2 = 0$, so that $\sigma^{-1} = \varepsilon\otimes\varepsilon - \zeta$. The resulting cocycle deformation

$$m_\sigma = (\sigma\otimes m\otimes\sigma^{-1})\Delta^{(2)}_{A\otimes A}\colon A\otimes A\to A$$

of the multiplication $m\colon A\otimes A \to A$ is then given by

$$m_\sigma = m + (\zeta\otimes m - m\otimes\zeta)\Delta_{A\otimes A} = \mu_0 + \mu_n$$

and is compatible with the original comultiplication. The explicit expression of m_σ in terms of the PBW-basis of A is

$$m_\sigma(x^ig^j \otimes x^kg^l) = q^{jk}(x^{i+k} + ax^\beta(1-g^{n\alpha}))g^{j+l},$$

where $i+j = n\alpha+\beta$ with $\alpha = 0, 1$. The identity $\sum_{s+v=\beta}\binom{i}{s}_q\binom{k}{v}_q q^{s(k-v)} = \binom{i+k}{\beta}$, which can be found in [Ka], has been used in the calculations. The deformed Hopf algebra has the presentation

$$A_\sigma = \langle x, g|\, x^n = a(1-g^n), g^{np} = 1, gx = qxg\rangle$$

with the original comultiplication, and since $\operatorname{gr}_c A_\sigma = A$ it is a lifting of A.

The linear dual $V^* = k\xi$, $\xi(x) = 1$, is a crossed module over the character group $\widehat{G} = \langle\theta\rangle$, $\theta(g) = \alpha$ a primitive np-th root of unity and $\alpha^p = q$, with action $\delta^*(\theta\otimes\xi) = \theta\xi = \alpha\xi$ and coaction $\mu^*(\xi) = \phi\otimes\xi$, where $\phi = \theta^p$. The graded braided Hopf algebra $\mathcal{A}(V^*) \cong k[\xi] \cong \mathcal{C}(V)^*$ is the graded polynomial algebra with comultiplication $\Delta(\xi^i) = \sum_{r+s=i}\binom{i}{r}_q\xi^r\otimes\xi^s$ so that ξ^n is primitive. The cofree graded braided Hopf algebra $\mathcal{C}(V^*) = k\,\langle\xi\rangle \cong \mathcal{A}(V)^*$ is the divided power Hopf algebra with basis $\{\,\xi_i|\, i\geq 0\}$, comultiplication $\Delta(\xi_i) = \sum_{r+s=i}\xi_r\otimes\xi_s$ and multiplication $\xi_i\xi_j = \binom{i+j}{i}_q\xi_{i+j}$. The quantum symmetrizer $\mathcal{S}\colon \mathcal{A}(V^*)\to\mathcal{C}(V^*)$ is given by $\mathcal{S}(\xi^i) = i_q!\xi_i$. The Nichols algebra of V^* is $B(V^*) = \mathcal{A}(V^*)/(\xi^n) \cong \operatorname{im}\mathcal{S}$ and the Hopf algebra

$$\begin{aligned} A^* &= B(V^*)\#k\widehat{G} \\ &= \langle\xi,\theta|\,\xi^n = 0, \theta^{np} = \varepsilon, \theta\xi = \alpha\xi\theta, \Delta(\xi) = \xi\otimes\varepsilon + \phi\otimes\xi, \Delta(\theta) = \theta\otimes\theta\rangle \end{aligned}$$

is radically graded.

The invertible element $\sigma^*\colon k \to A^*\otimes A^*$ with $\sigma^*(1) = \sigma = \varepsilon\otimes\varepsilon + \sum_{r+s=n} a_{rs}\xi^r\phi^s\otimes\xi^s = \varepsilon\otimes\varepsilon+\zeta$, with $a_{rs} = \frac{1}{r_q!s_q!}$, is the cocycle above represented in terms of the basis of A^*. Observe that ζ is of degree n and $\zeta^2 = 0$. The resulting cocycle deformation of the comultiplication

$$\Delta_\sigma = m^{(2)}_{A\otimes A}(\sigma\otimes\Delta\otimes\sigma^{-1} = \Delta + m_{A\otimes A}(\zeta\otimes\Delta - \Delta\otimes\zeta) = \delta_0 + \delta_n,$$

where $m^{(2)}_{H\otimes H}(\zeta\otimes\Delta\otimes\zeta) = 0$ is used, is compatible with the original multiplication. Since $\Delta(\theta)\zeta = \alpha^n\zeta\Delta(\theta)$, it follows that

$$\Delta_\sigma(\theta^i) = \theta^i\otimes\theta^i + (1-\alpha^{ni})\zeta(\theta^i\otimes\theta^i).$$

Using the identity $a_{u-1,s}q^s + a_{u,s-1} = a_{u,s}$ one finds that $\zeta\Delta(\xi) = \Delta(\xi)\zeta$, so that $\Delta_\sigma(\xi) = \Delta(\xi)$ and

$$\Delta_\sigma(\xi^i\theta^j) = \Delta(\xi^i)\Delta_\sigma(\theta^j).$$

The deformed Hopf algebra has the presentation

$$A^\sigma = \langle\xi,\theta|\,\Delta_\sigma(\xi) = \Delta(\xi), \Delta_\sigma(\theta) = \theta\otimes\theta + (1-\alpha^n)\zeta(\theta\otimes\theta)\rangle$$

with the original multiplication and radical filtration, so that $\operatorname{gr}_r A^\sigma = A^*$.

2. Let $G = \langle g\rangle$ be the cyclic group of order np_1p_2 and let α be a primitive root of unity of order np_1p_2. Consider the 1-dimensional crossed G-module $V = kx$ with action $gx = \alpha^{p_2}x$ and coaction $\delta(x) = g^{p_1} \otimes x$. The braiding $c\colon V \otimes V \to V \otimes V$ is then given by $c(x \otimes x) = g^{p_1}x \otimes x = \alpha^{p_1p_2}x \otimes x$.

The dual space $V^* = k\xi$ is a crossed module over the character group $\widehat{G} = \langle\theta\rangle$, where $\theta(g) = \alpha$. The action is given by

$$\delta^*(\theta \otimes \xi)(x) = (\theta \otimes \xi)\delta(x) = \theta(g^{p_1})\xi(x) = \alpha^{p_1}$$

and the coaction by

$$\mu^*(\xi(g^i \otimes x) = \xi(g^ix) = \alpha^{ip_2}\xi(x) = \alpha^{ip_2},$$

so that $\delta^*(\theta \otimes \xi) = \alpha^{p_1}\xi$ and $\mu^*(\xi) = \theta^{p_2} \otimes \xi$. The braiding map $c^*\colon V^* \otimes V^* \to V^* \otimes V^*$ is determined by $c^*(\xi \otimes \xi) = \theta^{p_2}\xi \otimes \xi = \alpha^{p_1p_2}\xi \otimes \xi$. This means that the dual V^* is obtained essentially by interchanging the role of p_1 and p_2, i.e: $(G, V, p_1, p_2)^* = (\widehat{G}, V^*, p_2, p_1)$.

The free graded braided Hopf algebra $\mathcal{A}(V) = k[x]$ is the polynomial algebra with comultiplication $\Delta(x^i) = (x \otimes 1 + 1 \otimes x)^i = \sum_{r+s=i} \binom{i}{r}_q x^r \otimes x^s$, where $q = \alpha^{p_1p_2}$. The ideal (x^n) is a Hopf ideal, since x^n is primitive. The cofree graded braided Hopf algebra $\mathcal{C}(V) = k\langle x\rangle$ is the divided power Hopf algebra with basis $\{x_i | \, i \geq 0\}$, comultiplication $\Delta(x_i) = \sum_{r+s=i} x_r \otimes x_s$ and multiplication $m(x_i \otimes x_j) = \binom{i+j}{i}_q x_{i+j}$. It follows in particular that $x_1^i = i_q! x_i$ and $x_1^n = n_q! x_n = 0$. The quantum symmetrizer $\mathcal{S}\colon \mathcal{A}(V) \to \mathcal{C}(V)$ is determined by $\mathcal{S}(x^i) = \mathcal{S}(x)^i = x_1^i = i_q! x_i$ and $\mathcal{S}(x^n) = N_q! x_n = 0$. The Nichols algebra of V is then $B(V) = \mathcal{A}(V)/(x^n) \cong \operatorname{im}\mathcal{S} \subset \mathcal{C}(V)$, and $\operatorname{im}\mathcal{S} = \oplus_{i=0}^{n-1} kx_i$ is the Hopf subalgebra of $\mathcal{C}(V)$ generated by x_1. The bosonization

$$A = B(V)\#kG = \langle g, x | \, g^{np_1p_2} = 1, x^n = 0, gx = \alpha^{p_2}xg, \Delta(x) = x \otimes 1 + g^{p_1} \otimes x\rangle$$

is coradically as well as radically graded, giving rise to liftings by deforming the multiplication and co-liftings by deforming the comultiplication.

The linear functional $\zeta\colon A \otimes A \to k$ of degree $-n$, defined by $\zeta(x^ig^j \otimes x^kg^l) = \alpha^{p_2jk}\delta_n^{i+k}$, is a Hochschild cocycle with $\zeta^2 = 0$, and satisfying

$$\zeta(m \otimes 1) * (\zeta \otimes \varepsilon) = \zeta(1 \otimes m) * (\varepsilon \otimes \zeta).$$

It follows that

$$\sigma = e^\zeta = \varepsilon \otimes \varepsilon + \zeta\colon A \otimes A \to k$$

is a convolution invertible multiplicative cocycle. In terms of the dual basis of A^* it can be expressed as

$$\sigma = \varepsilon \otimes \varepsilon + \zeta = \varepsilon \otimes \varepsilon + \sum_{r+s=n}^{0<r,s<n} a_{rs}\xi^r\theta^{p_2s} \otimes \xi^s,$$

where $a_{rs} = \frac{1}{r_q! s_q!}$. The corresponding cocycle deformation of the multiplication of A is

$$m_\sigma = (\sigma \otimes m \otimes \sigma^{-1})\Delta^{(2)}_{A\otimes A} = m + (\zeta \otimes m - m \otimes \zeta)\Delta_{A\otimes A},$$

since $(\zeta \otimes m \otimes \zeta)\delta_{A\otimes A} = 0$ (it is of degree $-2n$). Using

$$\Delta_{A\otimes A}(x^i g^j \otimes x^k g^l) = \sum_{r+s=i}^{u+v=k} \binom{i}{r}_q \binom{k}{u}_q x^r g^{p_1 s+j} \otimes x^u g^{p_1 v+l} \otimes x^s g^j \otimes x^v g^l$$

and invoking the identity [Ka]

$$\sum_{s+v=\beta} \binom{i}{s}_q \binom{k}{v}_q q^{s(k-v)} = \binom{i+k}{\beta} = \binom{n+\beta}{\beta} = 1$$

when $i + k = n + \beta$, the following explicit formula for m_σ can be deduced:

$$\begin{aligned} m_\sigma(x^i g^j \otimes x^k g^l) &= \alpha^{p_2 jk} x^{i+k} g^{j+l} \\ &+ \sum_{r+s=i}^{u+v=k} a \binom{i}{r}_q \binom{k}{u}_q \alpha^{p_2((p_1 s+j)u+jv)} (\delta_n^{r+u} x^{s+v} - x^{r+u} \delta_n^{s+v} g^{p_1(s+v)}) g^{j+l} \\ &= \alpha^{p_2 jk}[x^{i+k} + a x^\beta (\sum_{s+v=\beta} \binom{i}{s}_q \binom{k}{v}_q q^{s(k-v)} \\ &\quad - \sum_{r+u=\beta} \binom{i}{r}_q \binom{k}{u}_q q^{(i-r)u} g^{p_1 n}] g^{j+l} \\ &= \alpha^{p_2 jk}[x^{i+k} + a x^\beta (1 - g^{p_1 n})] g^{j+l}, \end{aligned}$$

where $i + k = n\gamma + \beta$ with $\gamma = 0, 1$.

The element $\zeta \colon k \to A \otimes A$, $\zeta(1) = \sum_{r+s=n}^{0<r,s<n} a_{rs} x^r g^{p_1 s} \otimes x^s$ is a Hochschild cocycle with $\zeta^2 = 0$. and satisfying

$$(\Delta \otimes 1)(\zeta)(\zeta \otimes 1) = (1 \otimes \Delta)(\zeta)(1 \otimes \zeta).$$

It then follows that $\sigma \colon k \to A \otimes A$, defined by

$$\sigma(1) = e^\zeta = 1 \otimes 1 + \zeta = 1 \otimes 1 + \sum_{r+s=n}^{0<r,s<n} a_{rs} x^r g^{p_1 s} \otimes x^s,$$

is invertible and satisfies the multiplicative 2-cocycle condition

$$(\Delta \otimes 1)(\sigma)(\sigma \otimes 1) = (1 \otimes \Delta)(\sigma)(1 \otimes \sigma).$$

The corresponding cocycle deformation of the comultiplication of A is

$$\begin{aligned} \Delta_\sigma &= m^{(2)}_{A\otimes A}(\sigma \otimes \Delta \otimes \sigma^{-1}) \\ &= \Delta + m_{H\otimes H}(\zeta \otimes \Delta - \Delta \otimes \zeta), \end{aligned}$$

since $m^{(2)}_{A\otimes A}(\zeta\otimes\Delta\otimes\zeta)=0$ (it is of degree $2n$). In the resulting Hopf algebra $(A,m,\Delta_\sigma,\iota,\varepsilon)$ we have

$$\Delta_\sigma(g)=g\otimes g+(1-\alpha^{p_2n})\zeta(g\otimes g)$$

and $\Delta_\sigma(g^{p_1})=g^{p_1}\otimes g^{p_1}=\Delta(g^{p_1})$. Moreover, a simple calculation using the identity $a_{u-1,s}q^s+a_{u.s-1}=a_{u,s}$ for $u+s=n+1$ shows that $\zeta\Delta(x)=\Delta(x)\zeta$, so that

$$\Delta_\sigma(x)=\Delta(x)+m_{A\otimes A}(\zeta\otimes\Delta-\Delta\otimes\zeta)(x)=\Delta(x).$$

3. If $g=\langle g\rangle$ is the cyclic group of odd prime order p and q is a primitive p-th root of unity, consider the 2-dimensional crossed G module $V=kx_1\oplus kx_2$ with action $gx_i=q^{(-1)^{i-1}}x_i$ and coaction $\delta(x_i)=g\otimes x_i$. The braiding map $c\colon V\otimes V\to V\otimes V$ is then $c(x_i\otimes x_j)=q^{(-1)^{j-1}}x_j\otimes x_i$.

The dual space $V^*=k\xi_1\oplus k\xi_2$ is a crossed module over the character group $\widehat{G}=\langle\theta\rangle$, where $\theta(g)=q$, with action $\theta\xi_i=q\xi_i$, coaction $\mu^*(\xi_i)=\theta^{(-1)^{i-1}}\otimes\xi_i$ and braiding $c(x_i\otimes x_j)=q^{(-1)^{j-1}}\xi_j\otimes\xi_i$.

The free graded braided Hopf algebra $\mathcal{A}(V)=T(V)$ is the tensor algebra with comultiplication determined by $\Delta(x_i)=x_i\otimes 1+1\otimes x_i$, where the braiding in the form of $\Delta m=(m\otimes m)(1\otimes c\otimes 1)(\Delta\otimes\Delta)$ has to be taken into account. In particular,

$$\begin{aligned}\Delta(x_i^p) &= (x_i\otimes 1+1\otimes x_i)^p=\sum_{r+s=p}\binom{p}{r}_q x_i^r\otimes x_i^s\\ &= x_i^p\otimes 1+1\otimes x_i^p,\end{aligned}$$

since $\binom{p}{r}_q=0$ for $0<r<p$, and

$$\begin{aligned}\Delta([x_1,x_2]_c) &= [x_1,x_2]_c\otimes 1+x_1\otimes x_2-c^2(x_1\otimes x_2)+1\otimes[x_1,x_2]\\ &= [x_1,x_2]_c\otimes 1+1\otimes[x_1,x_2],\end{aligned}$$

since $c^2(x_1\otimes x_2)=x_1\otimes x_2$, so that $[x_1,x_2]_c$ is primitive. The cofree graded braided Hopf algebra $\mathcal{C}(V)=k\langle x_1,x_2\rangle$ is the divided power Hopf algebra with basis all words in the variables $\left\{x_i^{(r)}\,\middle|\,i=0,1;r\geq 0\right\}$, comultiplication $\Delta(x_i^{(l)}=\sum_{r+s=l}x_i^{(r)}\otimes x_i^{(s)}$ and multiplication $m(x_i^{(r)}\otimes x_i^{(s)})=\binom{r+s}{r}_q x_i^{(r+s)}$ and $m(x_i^{1)}\otimes x_j^{(1)})=x_i^{(1)}x_j^{(1)}+q^{(-1)^{j-1}}x_j^{(1)}x_i^{(1)}$ if $i\neq j$. It follows in particular that $(x_i^{(1)})^r=r_q!x_i^{(r)}$, hence $(x_i^{(1)})^p=0$ and that $[x_i^{(1)},x_j^{(1)}]_c=0$ if $i\neq j$. The quantum symmetrizer $\mathcal{S}\colon\mathcal{A}(V)\to\mathcal{C}(V)$ is determined by $\mathcal{S}(x_ix_j)=x_i^{(1)}x_j^{(1)}+q^{(-1)^{j-1}}x_j^{(1)})x_i^{(1)}$ $\mathcal{S}(x_i^r)=\mathcal{S}(x_i)^r=(x_i^{(1)})^r=r_q!x_i^{(1)}$, so that $\mathcal{S}(x_i^p)=p_q!x_i^{(p)}=0$, and $\mathcal{S}([x_i,x_j]_c)=0$ for $i\neq j$. The Nichols algebra of V is then $B(V)=\mathcal{A}(V)/(x_1^p,x_2^p.[x_1,x_2]_c)\cong\operatorname{im}\mathcal{S}\subset\mathcal{C}(V)$, and $\operatorname{im}\mathcal{S}$ is the Hopf subalgebra of $\mathcal{C}(V)$ generated by $\left\{x_1^{(1)},x_2^{(1)}\right\}$. The bosonization

$$A=B(V)\#kG=\left\langle g,X_1,x_2\,\middle|\,g^p=1,x_1^p=0,x_2^p=0,x_1x_2=q^{-1}x_2x_1\right\rangle$$

with comultiplication $\Delta(g) = g \otimes g$ and $\Delta(x_i) = x_i \otimes 1 + g \otimes x_i$ is coradically graded.

The linear functional $\zeta\colon A \otimes A \to k$ of degree -2, defined by $\zeta(x_1^i x_2^j g^k \otimes x_1^r x_2^s g^t) = aq^k \delta_0^i \delta_1^j \delta_1^r \delta_0^s$, is a Hochschild cocycle with $\zeta^p = 0$

6. Duals of pointed Hopf algebras

6.1. **Liftings of Quantum linear spaces.** The duals of finite dimensional pointed Hopf algebras need not necessarily be pointed. Although the dual of the bicross product $E = \mathcal{B}(V)\#kG$, which is the bicross product $E^* = \mathcal{B}(W)\#k\widehat{G}$, is again pointed, the duals of its liftings are generally not pointed. We will explore the duals of such liftings H when V is a quantum linear space over a finite abelian group G. Then $V = \oplus_{i=1}^t kx_i$ with $x_i \in V_{g_i,\chi_i}$ with $\chi_i(g_j)\chi_j(g_i) = 1$. If $\chi_i(g_i)$ is a primitive n_i-th root of unity then $\dim \mathcal{B}(V) = n_1 n_2 ... n_t$ and by [Gr1] the finite dimensional liftings of E are of the form

$$H(a) = \langle G, V \,|\, gx_i = \chi_i(g)x_i g, [x_i, x_j] = a_{ij}(g_i g_j - 1), x_i^{n_i} = a_{ii}(g^{n_i} - 1)\rangle\,,$$

where $a_{ij} = 0$ when $g_i g_j = 1$ or $\chi_i \chi_j \neq \epsilon$ for $i \neq j$ and $a_{ii} = 0$ when $g_i^{n_i} = 1$ or $\chi_i^{n_i} \neq \epsilon$. Let G' be the subgroup of G generated by $\{\, g_i g_j, g_k^{n_k} \,|\, a_{ij} \neq 0, a_{kk} \neq 0\}$ and let $\bar{G} = G/G'$. Observe that the sequence of character groups

$$1 \to \widehat{G/G'} \to \widehat{G} \to \widehat{G'} \to 1$$

is exact, since k^* is divisible (k being algebraically closed). Then

$$A = H(a)/\left\langle [x_i, x_j], x_k^{n_k} \right\rangle \cong \mathcal{B}(V)\#k\bar{G}$$

fits into a commutative diagram

$$\begin{tikzcd} kG' \arrow[r] \arrow[d, equal] & kG \arrow[r] \arrow[d, "\kappa"'] & kG/G' \arrow[d, "\kappa"'] \\ kG' \arrow[r] & H \arrow[r] & A \end{tikzcd}$$

The map $\pi\colon H \to kG$, $\pi(x^m g) = \delta_{0m} g$, is a lifting of the canonical projection $\pi\colon A \to k\bar{G}$ and obviously satisfies $\pi\kappa = 1$. It is a coalgebra map, but not an algebra map if $a \neq 0$. Dualizing we get $A^* = \mathcal{B}(V^*)\#k\widehat{G/G'}$, a commutative diagram

$$\begin{tikzcd} A^* \arrow[r] \arrow[d, "\kappa^*"'] & H^* \arrow[r] \arrow[d, "\kappa^*"'] & k\widehat{G'} \arrow[d, equal] \\ k\widehat{G/G'} \arrow[r] & k\widehat{G} \arrow[r] & k\widehat{G'} \end{tikzcd}$$

and an algebra section $\pi^*\colon k\widehat{G} \to H^*$ for κ^*, $\pi^*(\chi)(x^m g) = \delta_{0m}\chi(g)$.

Proposition 6.1. *Let $V = \oplus_{i=1}^t kx_i$ be a quantum linear space over the finite abelian group G with $\chi_i(g_i)$ of order n_i in k^*, and let H be a non-trivial lifting of $E = \mathcal{B}(V)\#kG$. Then*

(1) $E^* \cong \mathcal{B}(V^*)\#k\widehat{G}$ *and*
(2) $G(H^*) = \widehat{G/G'}$ *is a proper subgroup of* $\widehat{G}$ *and* H^* *is not pointed.*

Proof. The dual $V^* = \oplus_{i=1}^t k\xi_i$, where $\xi_i(x_j) = \delta_{ij}$, is a crossed $k\widehat{G}$-module with action and coaction given by $\chi\xi_i = \chi(g_i)\xi_i$ and $\delta(\xi_i) = \chi_i \otimes \xi_i$. The commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}(V)^* & \xrightarrow{\mathcal{S}^*} & \mathcal{A}(V)^* \\ \cong\Big\downarrow & & \cong\Big\downarrow \\ \mathcal{A}(V^*) & \xrightarrow{\mathcal{S}} & \mathcal{C}(V^*) \end{array}$$

implies that $\mathcal{B}(V)^* \cong \mathcal{B}(V^*)$ as graded braided Hopf algebras over $k\widehat{G}$.

If $\chi \in G(H^*)$, i.e: $\chi\colon H \to k$ is an algebra map, then $\chi(g)\chi(x_i) = \chi(gx_i) = \chi_i(g)\chi(x_i g) = \chi_i(g)\chi(x_i)\chi(g)$, hence $\chi(x_i) = 0$ for all i. This implies that $0 = \chi(x_i^{n_i}) = a_{ii}\chi(g_i^{n_i} - 1)$ for all i and $0 = \chi([x_j, x_k]) = a_{jk}\chi(g_j g_k - 1)$ for all $j < k$, so that $\chi(G') = 1$. Thus $G(H^*) \subseteq \widehat{G/G'}$, and since $\widehat{G/G'} = G(A^*) \subseteq G(H^*)$, we conclude that $G(H^*) = \widehat{G/G'}$. Now

$$\dim \operatorname{Cor}(H^*) = \dim(H/\operatorname{Rad} H) = \dim H - \dim \operatorname{Rad} H \geq |G| = |\widehat{G}| > |G(H^*)|$$

implies that $\operatorname{Cor} H^*$ must contain a non-trivial matrix coalgebra component, i.e: that H^* is not pointed.

For a different proof observe that it suffices to show that $H/\operatorname{Rad} H$ is not commutative, since then $\operatorname{Cor}(H^*) \cong (H/\operatorname{Rad} H)^*$ and hence H^* is not pointed. First observe that $kG \cap \operatorname{Rad} H = 0$, since $\operatorname{Rad} H$ is nilpotent and $\operatorname{Rad} kG = 0$, so that

$$kG \xrightarrow{\kappa} H \xrightarrow{\eta} H/\operatorname{Rad} H$$

is injective. If $H/\operatorname{Rad} H$ were commutative then

$$0 = \eta(x_i)\eta(g) - \eta(g)\eta(x_i) = \eta(x_i g - g x_i) = (1 - \chi_i(g))\eta(x_i)\eta(g)$$

for $1 \leq i \leq t$ and every $g \in G$, and hence $x_i \in \operatorname{Rad} H$, since $\chi_i \neq 1$. This would imply that $x_i^{n_i} = a_{ii}(g_i^{n_i} - 1)$ and $[x_j, x_k] = a_{ij}(g_j g_k - 1)$ are in $\operatorname{Rad} H$ for all i and all $j < k$, respectively, contradicting $\operatorname{Rad} H \cap kG = 0$. □

6.2. **Examples.** Here are some examples of Hopf algebras with the property that $G(H^*) = \operatorname{Alg}(H, k)$ is trivial. Let G be a cyclic group of odd order n, $r > 1$ a divisor of n, q a primitive r-th root of unity and $H = H(a)$ any lifting of a quantum linear space over G defined by the generators g, x, y, the relations

$$g^n = 1, gx = qxg, gy = q^{-1}yg, [x, y] = c(g^2 - 1), x^r = a(g^r - 1), y^r = b(g^r - 1)$$

and comultiplication $\Delta(g) = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$, $\Delta(y) = y \otimes 1 + g \otimes y$. If $c \neq 0$ then $G' = \langle g^r, g^2 \rangle = G$ and hence $G(H^*) \cong \widehat{G/G'} = \{\varepsilon\}$.

1. The examples in [BDG] are of that form. If p is an odd prime number and q is a primitive p-th root of unity, then the Hopf algebra defined by generators g, x, y, relations

$$g^{p^2} = 1, gx = qxg, gy = q^{-1}yq, x^p = a(g^p - 1) = y^p, [x, y] = b(g^2 - 1)$$

and comultiplication $\Delta(g) = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$, $\Delta(y) = y \otimes 1 + g \otimes y$. Then $\dim H = p^4$ and $G(H^*)$ is trivial if $b \neq 0$.

2. If p is an odd prime number and q a primitive p-th root of unity then the algebra defined by generators g, x, y and relations

$$g^p = 1, gx = qxg, gy = q^{-1}yq, x^p = 0 = y^p, [x, y] = b(g^2 - 1)$$

is a Hopf algebra with comultiplication $\Delta(g) = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$, $\Delta(y) = y \otimes 1 + g \otimes y$. Moreover, $\dim H = p^3$ and $G(H^*)$ is trivial if $b \neq 0$.

This Hopf algebra also has an interesting property of having exactly p irreducible representations, one for each dimension between 1 and p. Indeed, assume that G, X, Y are $r \times r$ matrices ($r \geq 2$, it is clear the ε is the unique 1-dimensional representation), such that $g \mapsto G$, $x \mapsto X$ and $y \mapsto Y$ is an irreducible representation. Note that, if e is an eigenvector for G corresponding to an eigenvalue ψ, then either $X^iY^je = 0$ or X^iY^je is an eigenvector for G corresponding to an eigenvalue $\psi\xi^{i-j}$. Now choose an eigenvector e for G so that $Ye = 0$ and note that vectors $\mathcal{E} = (e, Xe, \ldots X^{r-1}e)$ must be a basis for k^n (since $\langle G, X, Y\rangle = \bigvee \left\{G^iX^jY^k\right\} = M_r(k)$ and $\bigvee X^ie = \bigvee G^iX^jY^ke$). In particular, this shows that $r \leq p$. In the ordered basis $\mathcal{E}$, the matrices G, X and Y are as follows.

$$G = \psi \begin{pmatrix} 1 & & & & & \\ & \xi^1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \xi^{r-l} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & y_1 & & & & \\ & 0 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & y_{r-1} \\ & & & & & 0 \end{pmatrix}.$$

Now, using the identity $XY - \xi^{-1}YX = G^2 - I$, it is easy to see that $\psi^2 = \xi^{1-r}$ and that

$$y_i = (\xi - \psi^2\xi^i)(\xi^i - 1)(\xi - 1)^{-1} = \xi(1 - \xi^{i-r})(\xi^i - 1)(\xi - 1)^{-1}.$$

It is also straightforward to check, that

$$(g, x, y) \mapsto (G, X, Y),$$

where G, X, Y are as above, is an irreducible representation of H.

3. Here is an example of even dimension. Let p be a prime number and let q be a primitive p-th root of unity. Define a Hopf algebra by generators g, x, y, relations

$$g^{2p} = 1, gx = qxg, gy = q^{-1}yq, x^p = a(g^p - 1) = y^p, [x, y] = b(g^2 - 1)$$

and comultiplication $\Delta(g) = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$, $\Delta(y) = y \otimes 1 + g \otimes y$. Then $\dim H = 2p^3$ and $G(H^*)$ is trivial if $a \neq 0 \neq b$.

4. Let $n = rs$ be a positive integer with $\gcd(r, s) = 1$. If $G = \langle g \rangle$ is a cyclic group of order n, then the character group $\widehat{G} = \langle \chi \rangle$ is also cyclic of order n. Let H be the Hopf algebra defined as an algebra by the generators g, x, y and the relations

$$\begin{aligned} g^n &= 1, gx = \chi^r(g)xg, gy = \chi^{-r}(g)yg, \\ x^r &= a(g^s - 1), y^r = c(g^s - 1), [x, y] = b(g^2 - 1), \end{aligned}$$

with comultiplication $\Delta(g) = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$, $\Delta(y) = y \otimes 1 + g \otimes y$. Then $\chi^r(g)$ and $\chi^{-r}(g)$ have order s. Moreover, $\dim H = ns^2$. Since $\gcd(r, s) = 1$, it follows that $G_0 = \left\langle g^s, g^2 \right\rangle$ if $a \neq 0 \neq c$, even if $b = 0$. Thus $G(H^*)$ is trivial, if s is odd, and of order 2 if s is even.

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